For any presentation ϕ and time t, define k-SPIA (Strictly k-Piecewise Inference Algorithm) as follows

$$k\text{-SPIA}\big(\varphi\langle t\rangle\big) = \begin{cases} \varnothing & \text{if } t = 0 \\ k\text{-SPIA}(\varphi\langle t-1\rangle) \cup \texttt{subseq}_k(\varphi(t)) & \text{otherwise} \end{cases}$$

Note that we are being a little sloppy here. Technically, the output of k-SPIA given some input sequence is a set of subsequences G, not a program. What we really mean with the above is that $k-$SPIA outputs a program which uses G to solve the membership problem for $L(G) = \{w \mid \texttt{subseq}_k(w) \subseteq G\}$. This program looks something like this.

1. Input: any word w.
2. Check whether $\texttt{subseq}_k(w) \subseteq G$.
3. If so, OUTPUT Yes, otherwise OUTPUT No.

All $k-$SPIA does is update this program simply by updating the contents of G.

Theorem 1. *For each k, $k-$SPIA identifies in the limit from positive data the collection of stringsets SP_k.*

Proof Consider any $k \in \mathbb{N}$. Consider any $S \in \text{SP}_k$. Consider any positive presentation φ for S. It is sufficient to show there exists a point in time t_ℓ such that for all $m \geq t_\ell$ the following holds:

1. k-SPIA$(\langle m\rangle) = k-$SPIA$(\langle t_\ell\rangle)$ (convergence), and
2. k-SPIA$(\langle m\rangle)$ is a program that solves the membership probem for S.

Since $S \in \text{SP}_k$, there is a finite set $G \subseteq \Sigma^{\leq k}$ such that $S = L(G)$.

Consider any subsequence $g \in G$. Since $g \in G$ there is some word $w \in S$ which contains g as a k-subsequence. Since G is finite, there are finitely many such w, one for each g in G. Because φ is a positive presentation for S, there is a time t where each of these w occurs. For each w let t be the first occurence of w in φ. Let t_ℓ denote the latest time point of all of these time points t. Next we argue that for all time points m larger than this t_ℓ, the output of $k-$SPIA correctly solves the membership problem for S and does not change.

Consider any $m \geq t_\ell$. The claim is that k-SPIA$(\langle m\rangle) = k-$SPIA$(\langle t_\ell\rangle) = G$. For each g in G, a word containing g as a subsequence occurs at or earlier than t_ℓ and so $g \in k-$SPIA$(\langle m\rangle)$. Since g was arbitrary in G, $G \subseteq k-$SPIA$(\langle m\rangle)$.

Similarly, for each $g \in k-$SPIA$(\langle m\rangle)$, there was some word w in φ such that w contains g as a subsequence. Since φ is a positive presentation for S, w is in S. Since w belongs to S, $\texttt{subseq}_k(w) \subseteq G$ and so g belongs to G. Since g was arbitrary in k-SPIA$(\langle m\rangle)$ it follows that k-SPIA$(\langle m\rangle) \subseteq G$.

It follows k-SPIA$(\langle m\rangle) = G$.

Since m was arbitrarily larger than t_ℓ we have both convergence and correctness.

Since φ was arbitrary for S, S arbitrary in SP_k and k arbitrary, the proof is concluded. □