

# Theoretical Computational Linguistics: Finite-state Automata

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# Chapter 1

## Introduction: Strings and Trees

### 1.1 Computational Linguistics: Course Overview

In this class, we will study:

1. Automata Theory
2. Haskell
3. ... as they pertain to problems in linguistics:
  - (a) *Well-formedness* of linguistic representations
  - (b) *Transformations* from one representation to another

#### 1.1.1 Linguistic Theory

Linguistic theory often distinguishes between *well- and ill-formed representations*.

**Strings.** In English, we can coin new words like *bling*. What about the following?

1. *gding*
2. *θwik*
3. *spif*

**Trees.** In English, we interpret the compound *deer-resistant* as an adjective, not a noun. What about the following?

1. *green-house*
2. *dry-clean*
3. *over-throw*

Linguistic theory is often also concerned with *transformations*.

**Strings.** In generative phonology, underlying representations of words are *transformed* to surface representations of words.

1. /kæt-z/ → [kæts]
2. /wɪf-z/ → [wɪfɪz]

**Trees.** In derivational theories of generative syntax, the deep sentence structure is *transformed* into a surface structure.

1. Mary won the competition.
  - (a) The competition was won by Mary.
  - (b) What did Mary win?

### 1.1.2 Automata Theory

Automata are *abstract* machines that answer questions like these.

#### The Membership Problem

**Given:** A possibly infinite set of strings (or trees)  $X$ .

**Input:** A input string (or tree)  $x$ .

**Problem:** Does  $x$  belong to  $X$ ?

#### The Transformation Problem

**Given:** A possible infinite function of strings to strings (or trees to trees)  $f : X \rightarrow Y$ .

**Input:** A input string (or tree)  $x$ .

**Problem:** What is  $f(x)$ ?

There are many kinds of automata. Two common types of automata address these specific problems.

**Recognizers** Recognizers solve the membership problem.

**Transducers** Transducers solve the transformation problem.

Different kinds of automata instantiate different kinds of memory.

**Finite-state Automata** An automata is finite-state whenever the amount of memory necessary to solve a problem for input  $x$  is fixed and **independent** of the size of  $x$ .

**Linear-bounded Automata** An automata is linear-bounded whenever the amount of memory necessary to solve a problem for input  $x$  is **bounded by a linear function** of the size of  $x$ .

In this class we will study finite-state recognizers and transducers. There are many types of these as well, some are shown below.

- deterministic vs. non-deterministic



- 1way vs. 2way (for strings)
- bottom-up vs. top-down vs. walking (for trees)

The simplest type is the deterministic, 1way recognizer for strings. We will start there and then complicate them bit by bit:

1. add non-determinism
2. add output (transducers)
3. add 2way-ness
4. generalize strings to trees and repeat

What do automata mean for linguistic theory?

**Fact 1:** Finite-state automata over strings are sufficient for phonology (Johnson, 1972; Kaplan and Kay, 1994).

**Fact 2:** Finite-state automata over strings are *NOT* sufficient for syntax, but linear-bounded automata are (Chomsky, 1956; Shieber, 1985, among others).

**Fact 3:** Finite-state automata over trees *ARE* sufficient for syntax (Rogers, 1998; Kobele, 2011; Graf, 2011).

**Hypothesis:** Linguistic phenomena can be modeled with special kinds of finite-state automata with even stricter memory requirements over the right representations (Graf, 2015; Heinz, forthcoming).

### 1.1.3 Haskell

As we learn automata theory in the context of linguistic problems, we will also learn to program in Haskell. Haskell...

1. is a functional programming language.
2. is fully typed.
3. uses lazy evaluation.
4. requires you to think different!

Specifically, we will implement automata and use them to model linguistic competence. When it comes to programming in Haskell, it is useful to keep some proverbs in mind.

1. You can lead a horse to water but you can't make him drink.
2. A teacher can open the door, but the student has to walk through it.

Programming can be joyous, but it can also be frustrating. It can be especially frustrating at first when everything is unfamiliar. Patience and persistence are very valuable when it comes to learning how to program.

Here is how to get started with Haskell. If your computer runs Windows, Mac, Linux, go to this website and install the Haskell Platform.

<https://www.haskell.org/platform/>

Alternatively, you can do everything online. If you use a Chromebook, then this probably your best option.

1. Create a free account here: <https://c9.io>
2. Create a new Ubuntu workspace.
3. Type the following in the command line:

```
sudo apt-get update
```

4. And then:

```
sudo apt-get install haskell-platform
```

### 1.1.4 HW for next time

1. Read Chapter 0 of *Introduction to the Theory of Computation* and do exercises 0.1-0.6 on page 13. The most important sections are on sets, sequences and tuples, functions and relations, strings and languages, and Boolean logic. Section 0.3 on proofs is good to read but you will not be writing proofs in this course. I do hope to go through some proofs, however.
2. Set up Haskell and start **ghci**. If you succeed, you should see something like this:

```
GHCi, version 7.6.3: http://www.haskell.org/ghc/  :? for help
Loading package ghc-prim ... linking ... done.
Loading package integer-gmp ... linking ... done.
Loading package base ... linking ... done.
Prelude>
```

You can type `:q` to exit **ghci**.

## 1.2 Strings and Trees

In this section, we define strings and trees of finite size inductively.

### 1.2.1 Strings

Informally, strings are sequences of symbols.

What are symbols? It is standard to assume a set of symbols called the *alphabet*. The Greek symbol  $\Sigma$  is often used to represent the alphabet but people also use  $S$ ,  $A$ , or anything else. The symbols can be anything: IPA letters, morphemes, words, part-of-speech categories.  $\Sigma$  can be infinite in size, but we will usually consider it to be finite.

Formally, strings are defined inductively with an operation called *concatenation*. Concatenation is an operation, like addition, which build new strings from existing ones. For now, we will write it as “.”. *Inductive definitions* are defined in two parts: the *base case* and an *inductive case*.

**Definition 1** (Strings, version 1).

**Base Case:** *If  $a \in \Sigma$  then  $a$  is a string.*

**Inductive Case:** *If  $a \in \Sigma$  and  $w$  is a string then  $(w) \cdot a$  is a string.*

**Example 1.** Let  $\Sigma = \{a, b, c\}$ . Then the following are strings.

1.  $((a) \cdot b) \cdot c$
2.  $((a) \cdot a) \cdot a$
3.  $((a) \cdot b) \cdot c \cdot c$

Frankly, writing all the parentheses and “.” is cumbersome. So the above examples are much more readable if written as follows.

1.  $abc$
2.  $aaa$
3.  $abcc$

Writing all the parentheses and “.” is also unnecessary because the above definition provides a unique “derivation” for each string.

**Example 2.** Let  $\Sigma = \{a, b, c\}$ . We claim  $w = (((a) \cdot b) \cdot c) \cdot c$  is a string. There is basically one way to show this. First we observe that  $c \in \Sigma$  so whether  $w$  is a string depends on whether  $x = ((a) \cdot b) \cdot c$  is a string by the inductive case. Next we observe that since  $c \in \Sigma$  whether  $x$  is a string depends on whether  $y = (a) \cdot b$  is a string, again by the inductive case. Once more, since  $b \in \Sigma$  whether  $y$  is a string depends on whether  $a$  is a string. Finally, by the base case  $a$  is a string and so the dominoes fall:  $y$  is a string so  $x$  is a string and so  $w$  is a string.

This unique derivability is useful in many ways. For instance, suppose we want to determine the length of a string. Here is how we can do it.

**Definition 2** (string length). *The length of a string  $w$ , written  $|w|$ , is defined as follows. If there is  $a \in \Sigma$  such that  $w = a$  then  $|w| = 1$ . If not, then  $w = (x) \cdot a$  where  $x$  is some string and  $a \in \Sigma$ . In this case,  $|w| = |x| + 1$ .*

Note length is an inductive definition!

**Example 3.** What is the length of string  $w = abcc$ ? Well, as before we see that  $w = xc$  where  $x = abc$ . Thus,  $|w| = |abc| + 1$ . What is the length of  $abc$ ? Well,  $x = yc$  where  $y = ab$ . So now we have  $|w| = (|ab| + 1) + 1$ . Since  $y = zb$  where  $z = a$  we have  $|w| = ((|a| + 1) + 1) + 1$ . Finally, by the *base case* we have  $|w| = (((1) + 1) + 1) + 1 = 4$ .

Finally, there is another way to define strings, which makes use of the so-called empty string. The empty string is usually written with one of the Greek letters  $\epsilon$  or  $\lambda$ . It's just a matter of personal preference. The empty string is useful from a mathematical perspective in the same way the number zero is useful. Zero is a special number because for all numbers  $x$  it is the case that  $0 + x = x + 0 = x$ . The empty string serves the same special purpose. It is the unique string with the following special property with respect to concatenation.<sup>1</sup>

$$\text{For all strings } w, \epsilon \cdot w = w \cdot \epsilon = w \tag{1.1}$$

With this concept, we can redefine strings as the following.

**Definition 3** (Strings, version 2).

**Base Case:**  $\epsilon$  is a string.

**Inductive Case:** If  $a \in \Sigma$  and  $w$  is a string then  $(w) \cdot a$  is a string.

As a technical matter, when we write the string  $abc$ , we literally mean the following structure:  $((\epsilon) \cdot a) \cdot b) \cdot c$ .

The definition for length can be redefined similarly.

**Exercise 1.** Informally, the *reverse* of a string writes the last letter first and so on. So the reverse of  $abcb$  is  $bcba$  and the reverse of  $aacbb$  is  $bbcaa$ . Write a formal definition of reverse.

The set of all strings, including the empty strings, is written  $\Sigma^*$ . A *stringset* is a subset of  $\Sigma^*$ .

Stringsets are often called *formal languages*. Formal language theory is the study of stringsets. From a linguistic perspective, it is the study of string well-formedness.

## 1.2.2 Trees

Trees are like strings in that they are recursive structures. Informally, trees are structures with a single ‘root’ which dominates a sequence of trees.

Formally, trees extend the dimensionality of string structures from 1 to 2. Like strings, we will assume an “empty” tree. Also like strings, we assume a set of symbols  $\Sigma$ . This is sometimes partitioned into symbols of different types depending on whether the symbols can only occur at the leaves of the trees or whether they can dominate non-empty trees. We don't make such a distinction here. In the following definition, parentheses are omitted for readability.

**Definition 4** (Trees).

**Base Case:**  $\tau$  is a tree, which we call the empty tree.

---

<sup>1</sup>Technically, concatenation is not yet defined between strings since it is only defined for a string and a symbol. Our goal would be to ensure the property mentioned holds once concatenation is defined between strings.

**Inductive Case:** If  $a \in \Sigma$  and  $t_1, t_2, \dots, t_n$  is a string of trees of length  $n$  then  $a[t_1 t_2 \dots t_n]$  is a tree.

Also, a tree  $a[\tau]$  is called a *leaf*.

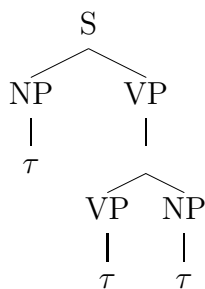
Here are some examples.

**Example 4.** Let  $\Sigma = \{\text{NP}, \text{VP}, \text{S}\}$ . Then the following are trees.

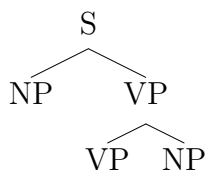
1.  $\text{S}[\text{NP}[\tau] \text{VP}[\text{VP}[\tau] \text{NP}[\tau]]]$
2.  $\text{NP}[\text{VP}[\tau] \text{S}[\tau] \text{S}[\tau] \text{VP}[\tau]]]$
3.  $\text{NP}[\text{NP}[\text{NP}[\tau] \text{VP}[\tau] \text{S}[\tau]]]$

We might draw these structures as follows. In (1b), (2) and (3), we omit the empty ones for readability.

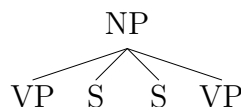
(1a)



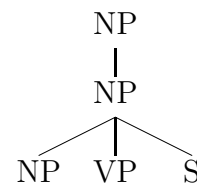
(1b)



(2)



(3)



Regarding the tree in (1), its leaves are NP, VP, and NP.

Note that the expression “a string of trees” in the definition of trees implies that our alphabet for strings is all the trees. Since we are defining trees, this may seem a bit circular. The key to resolving this circularity is to interleave the definition of the alphabet of the strings with the definition of trees in a zig-zag fashion. First we apply the inductive case for trees *once*, then we use those trees as an alphabet to define some strings of trees. Then we go back to trees and apply the inductive case again, which yields more trees which we can use to enlarge our set of strings and so on. We are not going to go through the technical details here.

As before, we can now write definitions to get information about trees. For instance here is a definition which gives us the number of non-empty nodes in the tree.

**Definition 5.** The size of a tree  $t$ , written  $|t|$ , is defined as follows. If  $t$  is empty, its size is 0. If not, then  $t = a[t_1 t_2 \dots t_n]$  where  $a \in \Sigma$  and each  $t_i$  is a tree. Then  $|t| = 1 + |t_1| + |t_2| + \dots + |t_n|$ .

**Exercise 2.** Using the above definition, calculate the size of the trees (1)-(3) above. Write out the calculation explicitly.

Here is a definition for the *depth* of a tree.

**Definition 6.** The depth of a tree  $t$ , written  $\mathbf{depth}(t)$ , is defined as follows. If  $t$  is empty, its depth is 0. If not, then  $t = a[t_1 t_2 \dots t_n]$  where  $a \in \Sigma$  and each  $t_i$  is a tree. Then  $\mathbf{depth}(t) = 1 + \max(\mathbf{depth}(t_1) + \mathbf{depth}(t_2) + \dots + \mathbf{depth}(t_n))$  where  $\max$  takes the largest number in the set.

**Exercise 3.** The *yield* of a tree  $t$ , written  $\mathbf{yield}(t)$ , maps a tree to a string of its leaves. For example let  $t$  be the tree in (1) in Example 4 above. Then its yield is the string “NP VP NP”.

### 1.2.3 String Exercises

**Exercise 4.** Let  $\Sigma$  be the set of natural numbers. So we are considering strings of numbers.

1. Write the definition of the *sum* of the numbers in the string. Using this definition, show how the sum of the following number strings is calculated.

(a)  $((\epsilon) \cdot 4) \cdot 11$

(b)  $((\epsilon) \cdot 1) \cdot 2) \cdot 3$

(c)  $\epsilon$

2. Write the definition of the *product* of the numbers in the string. Using this definition, show how the sum of the following number strings is calculated.

(a)  $((\epsilon) \cdot 4) \cdot 11$

(b)  $((\epsilon) \cdot 1) \cdot 2) \cdot 3$

(c)  $\epsilon$

**Exercise 5.** Let  $\Sigma$  be the set of natural numbers. So we are considering strings of numbers.

1. Write the definition of the function *addOne* which adds one to each number in the in the string. So *addOne* would change the string  $((\epsilon) \cdot 4) \cdot 11) \cdot 5$  to  $((\epsilon) \cdot 5) \cdot 12) \cdot 6$ . Using this definition, show *addOne* of the following number strings is calculated.

(a)  $((\epsilon) \cdot 4) \cdot 11$

(b)  $((\epsilon) \cdot 1) \cdot 2) \cdot 3$

(c)  $\epsilon$

2. Write the definition of the *timesTwo* of the numbers in the string. So *timesTwo* would change the string  $((\epsilon) \cdot 4) \cdot 11) \cdot 5$  to  $((\epsilon) \cdot 8) \cdot 22) \cdot 10$ . Using this definition, show *addOne* of the following number strings is calculated.

(a)  $((\epsilon) \cdot 4) \cdot 11$

(b)  $((\epsilon) \cdot 1) \cdot 2) \cdot 3$

(c)  $\epsilon$

### 1.2.4 Tree Exercises

**Exercise 6.** Let  $\Sigma$  be the set of natural numbers. Now let's consider trees of numbers.

1. Write the definition of the *sum* of the numbers in the tree. Using this definition, show how the sum of the following number trees is calculated. Note we are using an abbreviated notation.

(a)  $4[12\ 3[1\ 2]]$

(b)  $4[12[7\ 7[6]]\ 3[1\ 2]]$

(c)  $\tau$

2. Write the definition of the *yield* of the numbers in the string. The yield is a string with only the leaves of the tree in it. Using this definition, calculate the yields of the trees below.

(a)  $4[12\ 3[1\ 2]]$

(b)  $4[12[7\ 7[6]]\ 3[1\ 2]]$

(c)  $\tau$

**Exercise 7.** Let  $\Sigma$  be the set of natural numbers. Now let's consider trees of numbers.

1. Write the definition of the function *addOne* which adds one to each number in the tree. Using this definition, calculate *addOne* as applied to the trees below.

(a)  $4[12\ 3[1\ 2]]$

(b)  $4[12[7\ 7[6]]\ 3[1\ 2]]$

(c)  $\tau$

2. Write the definition of the function *isLeaf* which changes the nodes of a tree to **True** if it is a leaf node or to **False** if it is not. Using this definition, calculate *isLeaf* as applied to the trees below.

(a)  $4[12\ 3[1\ 2]]$

(b)  $4[12[7\ 7[6]]\ 3[1\ 2]]$

(c)  $\tau$





# Chapter 2

## String Acceptors

df

### 2.1 Deterministic Finite-state String Acceptors

#### 2.1.1 Orientation

This section is about deterministic finite-state acceptors for strings. The term *finite-state* means that the memory is bounded by a constant, no matter the size of the input to the machine. The term *deterministic* means there is single course of action the machine follows to compute the output from some input. As we will see later, *non-deterministic machines* can be thought of as pursuing multiple computations simultaneously. The term *acceptor* is synonymous with *recognizer*. It means that this machine solves *membership problems*: given a set of objects  $X$  and input object  $x$ , does  $x$  belong to  $X$ ? The term *string* means we are considering the membership problem over stringsets. So  $X$  is a set of strings (so  $X \subseteq \Sigma^*$ ) and the input  $x$  is a string.

#### 2.1.2 Definitions

**Definition 7.** A deterministic finite-state acceptor (DFA) is a tuple  $(Q, \Sigma, q_0, F, \delta)$  where

- $Q$  is a finite set of states;
- $\Sigma$  is a finite set of symbols (the alphabet);
- $q_0 \in Q$  is the initial state;
- $F \subseteq Q$  is a set of accepting (final) states; and
- $\delta$  is a function with domain  $Q \times \Sigma$  and co-domain  $Q$ . It is called the transition function.

We extend the domain of the transition function to  $Q \times \Sigma^*$  as follows. In these notes, the empty string is denoted with  $\lambda$ .

$$\begin{aligned}\delta^*(q, \lambda) &= q \\ \delta^*(q, aw) &= \delta^*(\delta(q, a), w)\end{aligned}\tag{2.1}$$

Consider some DFA  $A = (Q, \Sigma, q_0, F, \delta)$  and string  $w \in \Sigma^*$ . If  $\delta^*(q_0, w) \in F$  then we say  $A$  *accepts/recognizes/generates*  $w$ . Otherwise  $A$  *rejects*  $w$ .

**Definition 8.** *The stringset recognized by  $A$  is  $L(A) = \{w \in \Sigma^* \mid \delta^*(q_0, w) \in F\}$ .*

The use of the ‘L’ denotes “Language” as stringsets are traditionally referred to as *formal languages*.

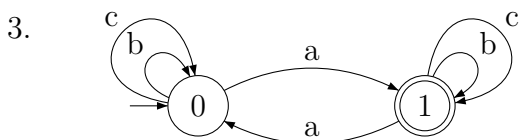
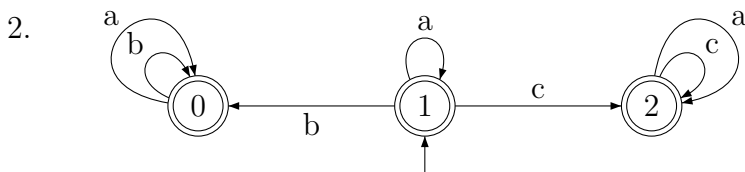
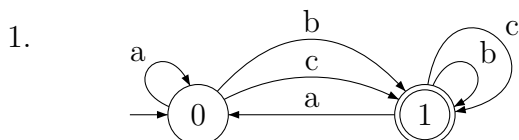
**Definition 9.** *A stringset is regular if there is a DFA that recognizes it.*

### 2.1.3 Exercises

**Exercise 8.** This exercise is about designing DFA. Let  $\Sigma = \{a, b, c\}$ . Write DFA which express the following generalizations on word well-formedness.

1. All words begin with a consonant, end with a vowel, and alternate consonants and vowels.
2. Words do not contain  $aaa$  as a substring.
3. If a word begins with  $a$ , it must end with  $c$ .
4. Words must contain two  $bs$ .
5. All words have an even number of vowels.

**Exercise 9.** This exercise is about reading and interpreting DFA. Provide generalizations in English prose which accurately describe the stringset these DFA describe.



4. Write the DFA in #1-3 in mathematical notation. So what is  $Q, \Sigma, q_0, F$ , and  $\delta$ ?

## 2.2 Properties of DFA

Note that for a DFA  $A$ , its transition function  $\delta$  may be partial. That is, there may be some  $q \in Q, a \in \Sigma$  such that  $\delta(q, a)$  is undefined. If  $\delta$  is a partial function,  $\delta^*$  will be also. It is assumed that if  $\delta^*(q_0, w)$  is undefined, then  $A$  rejects  $w$ .

We can always make  $\delta$  total by adding one more state to  $Q$ . To see how, call this new state  $\diamond$ . Then for each  $(q, a) \in Q \times \Sigma$  such that  $\delta(q, a)$  is undefined, define  $\delta(q, a)$  to equal  $\diamond$ . Every string which was formerly undefined w.r.t. to  $\delta^*$  is now mapped to  $\diamond$ , a non-accepting state. This state is sometimes called the *sink* state or the *dead* state.

**Definition 10.** A DFA is complete if  $\delta$  is a total function. Otherwise it is incomplete.

It is possible to write DFA which have many useless states. A state can be useless in two ways. First, there may be no string which forces the machine to transition into the state. Second, there may be a state from which no string

**Definition 11.** A state  $q$  in a DFA  $A$  is useful if there is a string  $w$  such that  $\delta^*(q_0, w) = q$  and a string  $v$  such that  $\delta^*(q, v) \in F$ . Otherwise  $q$  is useless. If every state in  $A$  is useful, then  $A$  is called trim.

Not all complete DFAs are trim. If there is a sink state, it is useless in the above sense of the word.

**Definition 12.** A DFA  $A$  is minimal if there is no other DFA  $A'$  such that  $L(A) = L(A')$  and  $A'$  has fewer states than  $A$ .

Technically, not all complete DFAs are minimal. If there is a sink state, it is not minimal.

**Exercise 10.** Consider the DFAs in the exercise 9. Are they complete? Trim? Minimal?

## 2.3 Some Closure Properties of Regular Languages

A set of objects  $X$  is *closed* under an operation  $\circ$  if for all objects  $x, y \in X$  it is the case that  $x \circ y \in X$  too.

We can easily show that the union of any two regular stringsets  $R$  and  $S$  is also regular. Let  $A_R = (Q_R, \Sigma, q_{0R}, F_R, \delta_R)$  be the DFA recognizing  $R$  and let  $A_S = (Q_S, \Sigma, q_{0S}, F_S, \delta_S)$  be the DFA recognizing  $S$ . We can assume  $A_R$  and  $A_S$  are complete. We assume the same alphabet.

Construct  $A = (Q, \Sigma, q_0, F, \delta)$  as follows.

- $Q = Q_R \times Q_S$ .
- $q_0 = (q_{0R}, q_{0S})$ .
- $F = \{(q_r, q_s) \mid q_r \in F_R \text{ or } q_s \in F_S\}$ .
- $\delta((q_r, q_s), a) = (q'_r, q'_s)$  where  $\delta_R(q_r, a) = q'_r$  and  $\delta_S(q_s, a) = q'_s$ .

**Theorem 1.**  $L(A) = R \cup S$ .

Similarly, the same kind of construction shows that the intersection of any two regular stringsets is regular. Construct  $B = (Q, \Sigma, q_0, F, \delta)$  as follows.

- $Q = Q_R \times Q_S$ .
- $q_0 = (q_{0R}, q_{0S})$ .
- $F = \{(q_r, q_s) \mid q_r \in F_R \text{ and } q_s \in F_S\}$ .
- $\delta((q_r, q_s), a) = (q'_r, q'_s)$  where  $\delta_R(q_r, a) = q'_r$  and  $\delta_S(q_s, a) = q'_s$ .

**Theorem 2.**  $L(B) = R \cap S$ .

Here are some additional questions we are interested in for regular stringsets  $R$  and  $S$ .

1. Is the complement of  $R$  (denoted  $\overline{R}$ ) a regular stringset?
2. Is  $R \setminus S$  a regular stringset?
3. Can we decide whether  $R \subseteq S$ ?
4. Is  $RS$  a regular stringset? (Note  $RS = \{rs \mid r \in R \text{ and } s \in S\}$ )
5. Is  $R^*$  a regular stringset? (Note  $R^0 = \{\lambda\}$ ,  $R^n = R^{n-1}R$ ,  $R^* = \bigcup_{n \in \mathbb{N}^0} R^n$ )

The answers to all of these questions is Yes. With a little thought about complete DFA, the answers to first three follow very easily.

**Theorem 3.** *If  $R$  is a regular stringset then the complement of  $R$  is regular.*

**Proof** (Sketch). If  $R$  is a regular stringset then there is a complete DFA  $A = (Q, \Sigma, q_0, F, \delta)$  which recognizes it. Let  $B = (Q, \Sigma, q_0, F', \delta)$  where  $F' = Q \setminus F$ . We claim  $L(B) = \overline{R}$ .  $\square$

**Corollary 1.** *If  $R, S$  are regular stringsets then so is  $R \setminus S$  since  $R \setminus S = R \cap \overline{S}$ .*

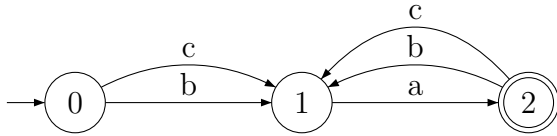
**Corollary 2.** *If  $R, S$  are regular stringsets then it is decidable whether  $R \subseteq S$  since  $R \subseteq S$  iff  $R \setminus S = \emptyset$ .*

**Corollary 3.** *If  $R, S$  are regular stringsets then it is decidable if  $R = S$  since  $R = S$  iff  $R \subseteq S$  and  $S \subseteq R$ .*

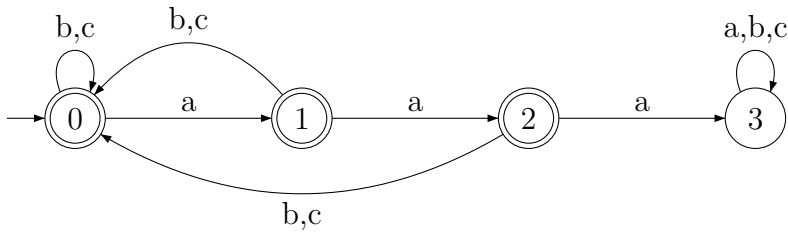
We postpone explaining how and why for the last two questions.

Some answers to Exercise 8

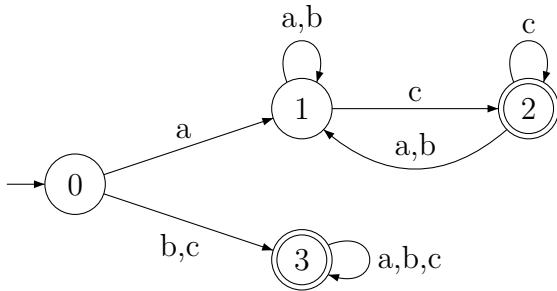
1.



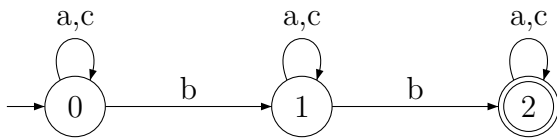
2.



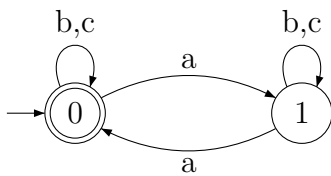
3.



4.



5.



Some answers to Exercise 9

1. Words must end in a consonant.
2. Words cannot contain both “b” and “c”. So words permit only one type of consonant.
3. Words must have an odd number of vowels.

## 2.4 Minimizing DFA

Here we show that for each regular stringset  $R$ , there is a smallest DFSA which recognizes  $R$ . This DFSA is unique (discounting the names of the states).

Consider any DFSA  $A$ . The idea is that some states are doing the “same work” as other states. Such states are said to be *indistinguishable*. States that are indistinguishable from each other are grouped into blocks. These blocks become the states of the minimal DFSA which recognizes the same stringset as  $A$ .

Given a DFSA  $A$  recognizing a stringset  $R$ , to find the minimal DFSA recognizing  $R$  we must do the following:

1. Determine which states of  $A$  are indistinguishable.
2. Using this information, construct the minimal DFSA.

### 2.4.1 Identifying indistinguishable states

Consider  $A = (Q, \Sigma, i, F, \delta)$ . Two states  $q, r$  are *distinguishable* in  $A$  if there is some string  $w$  such that  $\delta^*(q, w) \in F$  and  $\delta^*(r, w) \notin F$ . In other words, if there is a string that causes  $A$  to transition from state  $q$  to an accepting state but would cause  $A$  to transition from state  $r$  to a rejecting state then  $q$  and  $r$  are distinguishable. We say  $w$  *distinguishes*  $q$  from  $r$ .

We can determine whether a distinguishing string  $w$  exists for  $q$  and  $r$  recursively. There are two key observations to see why. First, observe that the empty string  $\lambda$  distinguishes accepting states from rejecting states. Second, if  $w$  distinguishes  $q$  from  $r$  and string  $u$  causes  $A$  to transition from state  $q'$  to  $q$  and causes  $A$  to transition from state  $r'$  to  $r$  then it is the case that string  $uw$  distinguishes  $q'$  from  $r'$ . Formally:

**Base case:**  $(q, r)$  is distinguishable if  $q \in F$  and  $r \notin F$ .

**Inductive Step:**  $(q', r')$  is distinguishable if  $(q, r)$  is distinguishable and there is  $a \in \Sigma$  such that  $\delta(q', a) = q$  and  $\delta(r', a) = r$ .

It is sufficient to check individual symbols in  $\Sigma$  in the inductive step and repeat it until no new distinguishable states are found.

To see why, consider the following. The first iteration checks to see if any 1-long strings distinguish states in  $A$ . The second iteration checks to see if any 2-long strings distinguish

states in  $A$ . Generally, the  $k$ th iteration checks to see if any  $k$ -long strings distinguish states in  $A$ . Importantly, if there is a string  $w$  of length  $k$  distinguishing  $q$  from  $r$  in  $A$  then there must be a string  $v$  of length  $k - 1$  and a symbol  $a \in \Sigma$  distinguishing  $q' = \delta(q, a)$  from  $r' = \delta(r, a)$  in  $A$ . This justifies why the inductive step can stop iterating if no new distinguishable states are found on the present iteration. It is not possible to find a  $k$ -long string distinguishing states if no  $(k - 1)$ -long string distinguishes any states.

At the end of this process we have a set of distinguishable state-pairs. The state-pairs in  $A$  that are not distinguishable are *indistinguishable*.

## 2.4.2 Building the minimal DFSA

Once the indistinguishable states have been identified, a new DFSA can be constructed. The indistinguishable state-pairs of  $A$  partition its states into *blocks*. A block is just a set of states. States  $q$  and  $r$  are in the same block only if  $(q, r)$  is an indistinguishable state-pair.

The process by which distinguishable states are found ensures that the set of indistinguishable state-pairs is closed under transitivity. In other words, if  $(q, r)$  and  $(r, s)$  are indistinguishable state-pairs then so is  $(q, s)$ . More generally, the *indistinguishable relation* is an equivalence relation satisfying not only transitivity but also reflexivity and symmetry.

Let  $B_q$  denote the block containing state  $q$ . Then the minimal DFSA  $M$  recognizing  $L(A)$  is given by:

- $Q_M = \{B_q \mid q \in Q\}$
- $i_M = B_i$
- $F_M = \{B_q \mid q \in F\}$
- $\delta_M(B, a) = B' \in Q_M$  such that  $B' \supseteq \{\delta(q, a) \mid q \in B\}$

## 2.4.3 Example

**Identifying indistinguishable pairs of states.** This example comes from (Hopcroft *et al.*, 2001, chapter 4). We are going to minimize the automaton shown below.

We need to identify distinguishable states. From the base case, each rejecting state is distinguishable from each accepting state with  $\lambda$ .

**Distinguishable pairs (Base Case):**

$$\{ (A,C), (B,C), (D,C), (E,C), (F,C), (G,C), (H,C) \}$$

A technical note: in addition to  $(A, C)$  the set should include  $(C, A)$  as well and similarly for the other pairs. However, we leave out these reflexive pairs for readability.

Next we repeatedly apply the inductive case. For each indistinguishable pair  $(q, r)$ , we ask is there  $a \in \Sigma$  such that  $(\delta(q, a), \delta(r, a))$  is distinguishable? If so, we add  $(q, r)$  to the list of

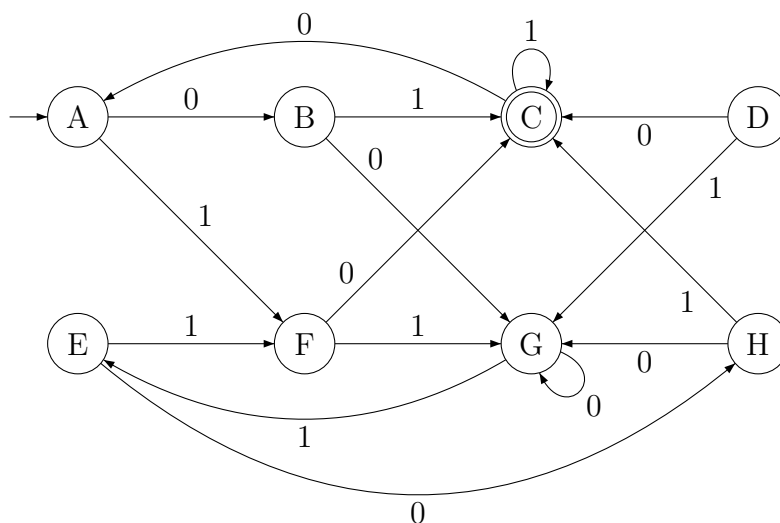


Figure 2.1: A non-minimal automaton (from Figure 4.8 in Hopcroft *et al.* (2001)).

distinguishable pairs. For example, consider the pair  $(A, B)$ . Since  $(\delta(A, 1), \delta(B, 1)) = (F, C)$  and  $(F, C)$  is distinguishable, we add  $(A, B)$  to the list. On the other hand  $(A, E)$  is not added to the list because neither  $(\delta(A, 1), \delta(E, 1)) = (F, F)$  nor  $(\delta(A, 0), \delta(E, 0)) = (B, H)$  are distinguishable states.

The added ones are shown in bold below.

### Distinguishable pairs (Inductive Step 1):

$$\left\{ \begin{array}{l} (A, C), (B, C), (D, C), (E, C), (F, C), (G, C), (H, C) \\ \mathbf{(A, B)}, \mathbf{(A, D)}, \mathbf{(A, F)}, \mathbf{(A, H)}, \mathbf{(B, D)}, \mathbf{(B, E)}, \mathbf{(B, F)}, \\ \mathbf{(B, G)}, \mathbf{(D, E)}, \mathbf{(D, G)}, \mathbf{(D, H)}, \mathbf{(E, F)}, \mathbf{(E, G)}, \mathbf{(E, H)}, \\ \mathbf{(F, G)}, \mathbf{(F, H)}, \mathbf{(G, H)} \end{array} \right\}$$

At this point, only four pairs of states are not yet known to be distinguishable. These are  $\{(A, E), (A, G), (B, H), (D, F)\}$ . We repeat the inductive step, to see if any new distinguished pairs are discovered. As before  $(A, E)$  is not found to be distinguishable. On the other hand,  $(A, G)$  is distinguishable now with string 1 and states  $F$  and  $E$  are now known to be distinguished. In fact, as a result of this step, only  $(A, G)$  is added as shown below.

### Distinguishable pairs (Inductive Step 2):

$$\left\{ \begin{array}{l} (A, C), (B, C), (D, C), (E, C), (F, C), (G, C), (H, C) \\ (A, B), (A, D), (A, F), (A, H), (B, D), (B, E), (B, F), \\ (B, G), (D, E), (D, G), (D, H), (E, F), (E, G), (E, H), \\ (F, G), (F, H), (G, H), \mathbf{(A, G)} \end{array} \right\}$$

The inductive step is repeated again, and this time no new distinguishable states are discovered. Therefore the iteration of the inductive steps terminates.



**Distinguishable pairs (Inductive Step 3):**

$$\left\{ \begin{array}{l} (A, C), (B, C), (D, C), (E, C), (F, C), (G, C), (H, C) \\ (A, B), (A, D), (A, F), (A, H), (B, D), (B, E), (B, F), \\ (B, G), (D, E), (D, G), (D, H), (E, F), (E, G), (E, H), \\ (F, G), (F, H), (G, H), (A, G) \end{array} \right\}$$

Thus, the only indistinguishable pairs are  $\{(A, E), (B, H), (D, F)\}$ .

**Building the minimal automaton.** With this information, the states are partitioned into blocks:

$$\left\{ \{A, E\}, \{B, H\}, \{C\}, \{D, F\}, \{G\} \right\}$$

These blocks are the states of the minimal automaton. Since block  $\{A, E\}$  contains the start state of the original acceptor, this block is the start state. Since block  $\{C\}$  contains a final state of the original acceptor, this block is a final state.

Finally, we calculate the transition function as shown in the diagram below. To illustrate, first Consider the transition from block  $\{A, E\}$  upon reading 1. With 1, the original delta function maps state  $A$  to  $F$  and  $E$  to  $F$ . There is exactly one block which contains  $F$ ; this is the block  $\{D, F\}$ . Hence the minimal machine transitions from state  $\{A, E\}$  to  $\{D, F\}$  with 1. The other transitions are determined similarly.

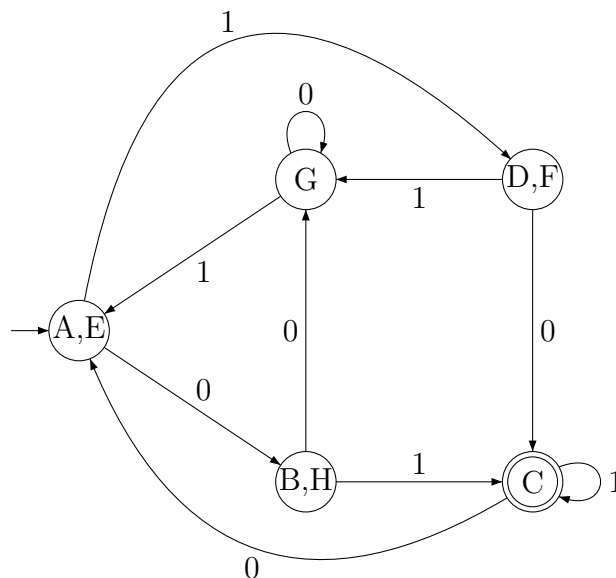


Figure 2.2: The minimal automaton recognizing the same stringset as the one in Figure 2.1.

## 2.5 Nonregular stringsets

Regular stringsets were defined as those subsets of  $\Sigma^*$  whose membership problem was solvable with a deterministic finite-state acceptor. It was also expressed that finite-state solvable membership problems invoke constant memory. This can be understood to mean that the number of states is constant and does not increase even as the words grow longer and longer.

In this section, we give a few examples of stringsets whose membership problem is not solvable with any DFSA. In each case, I hope to convey that in order to solve the membership problem for any word, the number of states needs to increase as words get longer.

**Example 5.** Let  $\Sigma = \{a, b, c\}$ .

1. The set of strings that with each string containing at least as many *as* as *bs*. Formally, we can define this set of strings as follows. For each  $a \in \Sigma$  and  $w \in \Sigma^*$ , let  $|w|_a$  be the number of times  $a$  occurs in  $w$ . So  $|abaa|_a = 4$  and  $|caca|_c = 2$  and so on. Then the set of strings we are interested in can be expressed as  $S_1 = \{w \mid |w|_a \geq |w|_b\}$ .
2. The palindrome language. Recall that a palindrome is a word that is the same when written both forwards and backwards. So the palindrome language contains all and only those words that are palindromes. Formally, we can define this set of strings as follows. Let us write  $w^R$  to express the reverse of  $w$ . So  $abac^R = caba$ . Then this stringset can be expressed as  $S_2 = \{wxw^R \mid w \in \Sigma^*, x \in \{\lambda, a, b, c\}\}$ .
3. The  $a^n b^n$  stringset. This is pronounced “ $a$  to the  $n$ ,  $b$  to the  $n$ ”. Recall that  $a^n$  defines the string with  $a$  concatenated to itself  $n$  times. So  $a^4 = aaaa$  and  $c^3 = ccc$  and so on. Note  $a^0 = \lambda$ . So this stringset can be expressed as  $S_3 = \{a^n b^n \mid 0 \leq n\}$ .

### 2.5.1 Exercises

**Exercise 11.** For each of the above examples, we try to write a deterministic finite-state acceptor. This exercise will help us understand why it is impossible.

1. First write an acceptor for  $S_1$  for words up to size 3. Next try to write the acceptor for words up to size 5. What is happening? What information is each state keeping tracking of?
2. First write an acceptor for  $S_2$  for words up to size 2. Next try to write the acceptor for words up to size 4. What is happening? What information is each state keeping tracking of?
3. First write an acceptor for  $S_3$  for words up to size 2. Next try to write the acceptor for words up to size 4. What about up to size 6? What is happening? What information is each state keeping tracking of?

In each case, the information that the states need to keep track of grows as words get longer. This is the basic insight into why the problem of these stringsets (and many like them) cannot be solved with finite-state acceptors.

## 2.5.2 Formal Analysis

Formal proofs that these are nonregular exist, based on abstract properties of regular stringsets as discussed in (Sipser, 1997; Hopcroft *et al.*, 2001) and elsewhere.

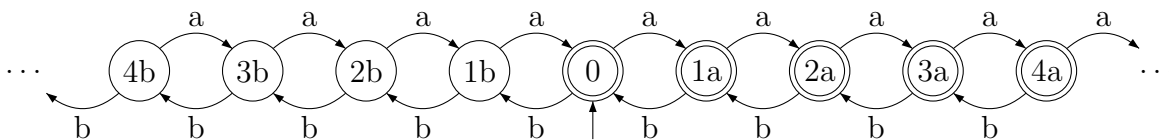
Here is one way to provide a rigorous proof with what we have learned so far.

Recall that when minimizing a DFSA  $A$ , we had to determine whether two states did the “same work” or not. We said states did different work, if there was a string that *distinguished* them. A string *distinguishes* state  $q$  from  $r$  if  $A$  would transition to a final state from  $q$  but to a non-final state from  $r$  (or vice versa). We can use distinguishing strings to see that a DFSA for the above stringsets would have infinitely many states.

Suppose there was a DFSA  $A$  for  $S_1$ . Let  $q_0$  be the initial state of  $A$  and let  $q_a = \delta(q_0, a)$ . Now we can ask are there any strings which distinguish  $q$  from  $q_a$ ? The answer is Yes. The string  $b$  distinguishes them because  $A$  must transition to an accepting state from  $q_a$  with  $b$ , but must transition to a non-accepting state from  $q_0$  with  $b$ . Thus  $q_0$  and  $q_a$  are distinct states.

We can repeat this reasoning. Let  $q_{aa} = \delta(q_a, a)$ . Is there a string which distinguishes  $q_{aa}$  from  $q_a$ ? Yes, in fact the string  $bb$  distinguishes them because  $A$  must transition to an accepting state from  $q_{aa}$  with  $bb$ , but must transition to a non-accepting state from  $q_a$  with  $bb$ . So  $q_a$  and  $q_{aa}$  are distinct states. What about  $q_{aa}$  and  $q_0$ ? They are distinct states too as witnessed by the distinguishing string  $b$ . So now we have three distinct states.

More generally, define state  $q_i$  to be  $\delta^*(q_0, a^i)$ . For any numbers  $n, m$  with  $n > m$ , one can show that  $q_n$  is a distinct state from  $q_m$ . We have to find a distinguishing string for these two states. The string  $b^n$  is such a string. The DFSA  $A$  must transition to an accepting state from  $q_n$  with  $b^n$  because the word  $a^n b^n$  has at least as many  $a$ s as  $b$ s. However,  $A$  transitions to a non-accepting state from  $q_m$  with  $b^n$  because the word  $a^m b^n$  has more  $b$ s as  $a$ s since  $n > m$ . Thus  $A$  must have infinitely many states if it is to accept words of arbitrary length. And we haven’t even looked at words beginning with  $bs$  yet!



**Exercise 12.** Present an argument like the one above for  $S_2$  and  $S_3$ .



# Chapter 3

## String Transducers

### 3.1 Deterministic Finite-state String Transducers

#### 3.1.1 Orientation

This section is about deterministic finite-state transducers for strings. The term *finite-state* means that the memory is bounded by a constant, no matter the size of the input to the machine. The term *deterministic* means there is single course of action the machine follows to compute the output from some input. The term *transducer* means this machine solves *transformation problems*: given an input object  $x$ , what object  $y$  is  $x$  transformed into? The term *string* means we are considering the transformation problem from strings to objects. So  $x$  is a string. As we will see, we can easily write transducers where  $y$  is a string, natural number, real number, or even a finite stringset! We will also see that DFSAs are a specific case of DFSTs.

However, we first define the output of the transformation to be a string. Then we will generalize it.

#### 3.1.2 Definitions

**Definition 13.** A deterministic finite-state string-to-string transducer (DFST) is a tuple  $T = (Q, \Sigma, M, q_0, v_0, \delta, \Omega, F)$  where

- $Q$  is a finite set of states;
- $\Sigma$  is a finite set of symbols (the input alphabet);
- $\mu$  is a finite set of output symbols (the output alphabet);
- $q_0 \in Q$  is the initial state;
- $v_0 \in \mu$  is the initial value;
- $\delta$  is a function with domain  $Q \times \Sigma$  and co-domain  $Q$ . It is called the transition function.
- $\Omega$  is a function with domain  $Q \times \Sigma$  and co-domain  $\mu$ . Let's call it the output function.
- $F$  is a function with domain  $Q$  and co-domain  $\mu$ . Let's call it the final function.

As before we generalize the  $\delta$  function to  $\delta^* : Q \times \Sigma^* \rightarrow Q$ . We also define a new function “process”  $\pi : Q \times \mu^* \times \Sigma^* \rightarrow Q \times \mu^*$  as follows.  $\pi$  will process the string and produce its output.

$$\begin{aligned}\pi(q, v, \lambda) &= (q, v \cdot F(q)) \\ \pi^*(q, v, aw) &= \pi((\delta(q, a), v \cdot \Omega(q, a), w))\end{aligned}\tag{3.1}$$

Consider some DFST  $T = (Q, \Sigma, q_0, F, \delta)$  and string  $u \in \Sigma^*$ . Then  $T(u) = v$  where  $(q, v) = \pi(q_0, v_0, u)$ . We say  $T$  transforms  $u$  into  $v$ .

**Definition 14.** The function defined by  $T$  is  $\{(u, v) \mid \exists q \in Q \text{ such that } \pi(q_0, v_0, u) = (q, v)\}$ . We write  $T(u) = v$  iff  $(u, v) \in T$ .

**Definition 15.** A string-to-string function is called sequential if there is a DFST that recognizes it.

### 3.1.3 Exercises

**Exercise 13.** Let  $\Sigma = \mu = \{a, e, i, o, u, p, t, k, b, d, g, m, n, s, z, l, r\}$ .

1. Write a transducer that prefixes  $pa$  to all words.
2. Write a transducer that suffixes  $ing$  to all words.
3. Write a transducer that deletes word initial vowels.
4. Write a transducer that voices obstruents which occur immediately after nasals.
5. Write a transducer that deletes word final vowels. So  $T(abba) = abb$  and  $T(pie) = pi$ .
6. Write a transducer that voices obstruents intervocalically.

Note that the transition function and the final function can be partial functions. In this case, the transducer is *incomplete* in the sense it is not defined for all inputs. As before, we will strive to make our sequential transducers describe total functions.

## 3.2 Some Closure Properties of Sequential functions

**Theorem 4** (Closure under composition). *If  $f, g$  are sequential functions then so is  $f \circ g$ .*

**Theorem 5** (Closure under intersection). *If  $f, g$  are sequential functions then so is  $f \cap g$ .*

**Theorem 6** (Minimal canonical form). *For every sequential string-to-string function  $f$ , it is possible to compute a DFST  $T$  such that  $T$  is equivalent to  $f$  and no other DFST  $T'$  equivalent to  $f$  has fewer states than  $T$ .*

The closure theorems follow from proofs which use the product construction. The minimal canonical form result is due to Choffrut (see his 2003 survey).

Sequential functions are not closed under union. This is because they are functions and so each input has a unique output.

### 3.3 Generalizing sequential functions with monoids

A *monoid* is a mathematical term which means any set which is closed under some associative binary operation with an identity. So if  $(S, *, 1)$  is a monoid then for all  $x, y \in S$ :

1. is the case that  $x * y$  is in  $S$  too (Closure under  $*$ )
2.  $(x * y) * z = x * (y * z)$  (Associativity)
3.  $1 * x = x * 1 = x$  (1 is the identity)

It is typical to refer to  $*$  as “times”, “multiplication” or as a product. It is also typical to refer to 1 as the “identity”, “unit” or “one”.

$\Sigma^*$  is closed under the binary operation of concatenation. Also the empty string behaves like the identity with respect to concatenation. So  $(\Sigma^*, \cdot, \lambda)$  is a monoid. As we processed the input string, we moved from state to state and updated the output value by concatenating strings along the output transitions. We can do the same thing and update the output value using some other product from another monoid.

**Example 6.** Here are some examples.

1.  $(\{\text{True}, \text{False}\}, \wedge, \text{True})$ . Boolean values and conjunction. This monoid shows the membership problem is a special case of the transformation problem.
2.  $(\mathbb{N}, +, 0)$ . Natural numbers and addition. Useful for counting!
3.  $([0, 1], \times, 1)$ . The real unit interval and multiplication. Useful for probabilities!
4.  $(\mathbb{R}, \times, 1)$ . All real numbers and multiplication.
5.  $(\text{FIN}, \cdot, \{\lambda\})$  where FIN is the class of finite stringsets and  $(\cdot)$  is concatenation of stringsets.
  - $\text{FIN} = \{S \mid \exists n \in \mathbb{N} \text{ with } |S| = n\}$
  - $S_1 \cdot S_2 = \{u \cdot v \mid u \in S_1, v \in S_2\}$ .
6. There are many others!

This means we can generalize DFSTs to transducers which output elements from any monoid.

**Definition 16.** A generalized sequential transducer (GST) is a tuple  $T = (Q, \Sigma, M, q_0, v_0, \delta, \Omega, F)$  where

- $Q$  is a finite set of states;
- $\Sigma$  is a finite set of symbols (the input alphabet);
- $(\mu, *, 1)$  is a monoid
- $q_0 \in Q$  is the initial state;
- $v_0 \in \mu$  is the initial value;
- $\delta$  is a function with domain  $Q \times \Sigma$  and co-domain  $Q$ . It is called the transition function.

- $\Omega$  is a function with domain  $Q \times \Sigma$  and co-domain  $\mu$ . Let's call it the output function.
- $F$  is a function with domain  $Q$  and co-domain  $\mu$ . Let's call it the final function.

The process function  $\pi$  is the same except we replace concatenation of the outputs with the monoid operator  $*$ .

$$\begin{aligned}\pi(q, v, \lambda) &= (q, v * F(q)) \\ \pi^*(q, v, aw) &= \pi((\delta(q, a), v * \Omega(q, a), w))\end{aligned}\tag{3.2}$$

Then, the definition of the function computed by the transducer is identical to what was written formerly.

That's it!

### 3.3.1 Exercises

**Exercise 14.** Let  $\Sigma = \mu = \{a, e, i, o, u, p, t, k, b, d, g, m, n, s, z, l, r\}$ .

1. Write a transducer that counts how many NC (nasal-consonant) sequences occurs in the input word.
2. Write a transducer that optionally voices obstruents which occur immediately after nasals. So for in input like *anta* the output should be the set  $\{anda, anta\}$ . (Hint: use the FIN monoid).

## 3.4 Learning more

Unfortunately, the material on deterministic transducers has yet to make its way into standard textbooks. Standard textbooks in computer science discuss non-deterministic finite-state transducers if they discuss transducers at all. Within computational linguistics where transducers are widely used, non-deterministic ones are the norm. See, for instance Roche and Schabes (1997); Beesley and Karttunen (2003) and Jurafsky and Martin (2008a). A notable exception is work by Mehryar Mohri (1997; 2005). The most textbook-like discussion of sequential functions I am aware of comes from Lothaire (2005, chapter 1).

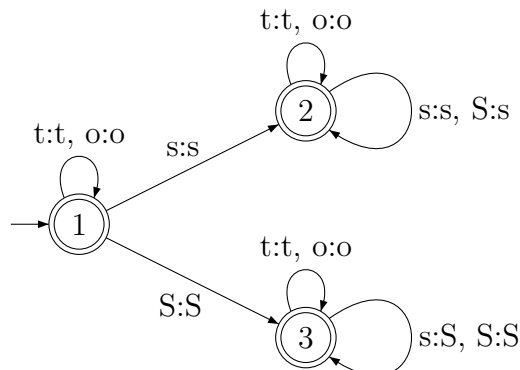
My interest in sequential transducers stems from three interrelated facts. First, they appear sufficient to describe morpho-phonological generalizations in natural language. So the extra power that comes with non-deterministic transducers appears unnecessary in this domain. Second, sequential transducers have canonical forms (Choffrut's theorem), but non-deterministic ones do not. Third, the class of sequential transducers can be learned from examples, unlike the class of non-deterministic transducers (de la Higuera, 2010, chapter 18).

## 3.5 Left and Right Sequential Transducers

Below is a sequential transducer for progressive sibilant harmony. This means later sibilants agree in anteriority with the first sibilant in the word. The alphabet here is simply  $\{s, S, t, o\}$



with  $\{s,S\}$  signifying the two classes of sibilants and  $\{t\}$  other consonants, and  $\{o\}$  the vowels. We assume the initial value is  $\lambda$  and for all  $q$  the final function maps  $q$  to  $\lambda$ .



Many examples of sibilant harmony in natural language are not progressive. They are in fact regressive. For instance here are some wonderfully long words in Samala (Chumash) (Applegate 1972). The underlying form is on the left and the surface form is on the right. There are some alternations in the vowels which we ignore here.

/ha-s-xintila-waf/	[haʃxintilawaf]	‘his former Indian name’
/k-su-kili-mekeken-f/	[kʃukʻilimeketʃ]	‘I straighten myself up’
/k-su-al-puj-un-faʃi/	[kʃalpujatʃiʃi]	‘I get myself wet’
/s-taja-nowon-waf/	[ʃtojowonowaf]	‘it stood upright’

**Exercise 15.** Explain why regressive sibilant harmony *cannot* be modeled with sequential transducers.

There is a straightforward way to address this issue. The transducers we are using process strings from left-to-right. However, transducers could also process strings from right-to-left. If we use the transducer above to process the Samala strings from right to left, we can model regressive sibilant harmony. The example below shows  $/stototooS/ \mapsto [StototooS]$ .

	s	t	o	t	o	t	o	o	S					
←	3	←	3	←	3	←	3	←	3	←	3	←	1	←
	S	t	o	t	o	t	o	o	S					

If a transducer processes the string left-to-right, it is called a *left sequential* transducer. If it processes it right-to-left it is called a *right sequential* transducer.

**Theorem 7.** *Left sequential functions and right sequential functions are incomparable; that is, there are functions that are both left and right sequential; neither left nor right sequential; left but not right sequential; and right but not left sequential.*

Progressive sibilant harmony is a case in point. It is left sequential but not right sequential. On the other hand, regressive sibilant harmony is right sequential, but not left

sequential. The identity function is both left and right sequential. Can you think of a function which is neither left nor right sequential?

The Haskell implementation of strings is inherently left-to-right because the outermost element in the list structure is on the left. If we were to define lists so that the outermost element of the list structure was on the right, it would become natural to process strings right-to-left.

An easy way to simulate a right sequential transducer with the lists we have in Haskell is to do the following.

1. Implement left sequential transducers as we have done.
2. Before processing a string  $w$ , reverse it.
3. Then reverse the output of the transducer.

In other words, `transduce t w` will process the string left-to-right. However, the function `reverse (transduce t (reverse w))` will simulate  $t$  processing  $w$  right-to-left.

# Chapter 4

## Tree Acceptors

### 4.1 Deterministic Bottom-up Finite-state Tree Acceptors

#### 4.1.1 Orientation

This section is about deterministic bottom-up finite-state tree acceptors. The term *finite-state* means that the memory is bounded by a constant, no matter the size of the input to the machine. The term *deterministic* means there is single course of action the machine follows to compute its output. The term *acceptor* means this machine solves *membership problem*: given a set of objects  $X$  and input object  $x$ , does  $x$  belong to  $X$ ? The term *tree* means we are considering the membership problem over *treesets*. The term *bottom-up* means that for each node  $a$  in a tree, the computation solves the problem by assigning states to the children of  $a$  before assigning states to  $a$  itself. This contrasts with *top-down* machines which assign states to  $a$  first and then the children of  $a$ . Visually, these terms make sense provided the root of the tree is at the top and branches of the tree move downward.

*Acceptor* is synonymous with *recognizer*. *Treeset* is synonymous with *tree language*.

A definitive reference for finite-state automata for trees is freely available online. It is “Tree Automata Techniques and Applications” (TATA) (Comon *et al.*, 2007). The presentation here differs from the one there, as mentioned below.

#### 4.1.2 Definitions

We will use the following definition of trees.

**Definition 17** (Trees). *We assume an alphabet  $\Sigma$  and symbols  $[ ]$  not belonging to  $\Sigma$ .*

**Base Cases:** *For each  $a \in \Sigma$ ,  $a[ ]$  is a tree.*

**Inductive Case:** *If  $a \in \Sigma$  and  $t_1 t_2 \dots t_n$  is a string of trees of length  $n$  then  $a[t_1 t_2 \dots t_n]$  is a tree.*

Let  $\Sigma^T$  denote the set of all trees of finite size using  $\Sigma$ . We also write  $a[\lambda]$  for  $a[ ]$ .

This differs from our previous definition in that we have excluded the empty tree. Instead, the leaves of trees are the base cases.

It will be helpful to review the following concepts related to functions: *domain*, *co-domain*, *image*, and *pre-image*. A function  $f : X \rightarrow Y$  is said to have domain  $X$  and co-domain  $Y$ . This means that if  $f(x)$  is defined, we know  $x \in X$  and  $f(x) \in Y$ . However,  $f$  may not be defined for all  $x \in X$ . Also,  $f$  may not be *onto*  $Y$ , there may be some elements in  $Y$  that are never “reached” by  $f$ .

This is where the concepts *image* and *pre-image* come into play. The image of  $f$  is the set  $\{f(x) \in Y \mid x \in X, f(x) \text{ is defined}\}$ . The pre-image of  $f$  is the set  $\{x \in X \mid f(x) \text{ is defined}\}$ . So the pre-image of  $f$  is the subset of the domain of  $f$  where  $f$  is defined. The image of  $f$  is the corresponding subset of the co-domain of  $f$ .

With this in place, we can define our first tree acceptor.

**Definition 18** (DBFTA). A Deterministic Bottom-up Finite-state Acceptor (DBFTA) is a tuple  $(Q, \Sigma_r, F, \delta)$  where

- $Q$  is a finite set of states;
- $\Sigma$  is a finite alphabet;
- $F \subseteq Q$  is a set of accepting (final) states; and
- $\delta : Q^* \times \Sigma \rightarrow Q$  is the transition function. The pre-image of  $\delta$  must be finite. This means we can write it down—for example, as a list.

We use the transition function  $\delta$  to define a new function  $\delta^* : \Sigma^T \rightarrow Q$  as follows.

$$\begin{aligned} \delta^*(a[\lambda]) &= \delta(\lambda, a) \\ \delta^*(a[t_1 \cdots t_n]) &= \delta(\delta^*(t_1) \cdots \delta^*(t_n), a) \end{aligned} \tag{4.1}$$

There are some important consequences to the formulation of  $\delta^*$  shown here. One is that  $\delta^*$  is undefined on tree  $a[t_1 \cdots t_n]$  if no transition  $\delta(q_1 \cdots q_n, a)$  is defined.

(Also, I am abusing notation since  $\delta^*$  is strictly speaking not the transitive closure of  $\delta$ .)

**Definition 19** (Treeset of a DBFTA). Consider some DBFTA  $A = (Q, \Sigma, F, \delta)$  and tree  $t \in \Sigma^T$ . If  $\delta^*(t)$  is defined and belongs to  $F$  then we say  $A$  accepts/recognizes  $t$ . Otherwise  $A$  rejects  $t$ . The treeset recognized by  $A$  is  $L(A) = \{t \in \Sigma^T \mid \delta^*(t) \in F\}$ .

The use of the ‘L’ denotes “Language” as treesets are traditionally referred to as *formal tree languages*.

**Definition 20** (Recognizable Treesets). A treeset is recognizable if there is a DBFTA that recognizes it.

### 4.1.3 Notes on Definitions

We have departed a bit from standard definitions. In particular, most introductions to tree automata make use of a particular kind of alphabet called a *ranked alphabet*. A ranked alphabet  $\Sigma_r$  is an alphabet  $\Sigma$  with an arity function  $ar : \Sigma \rightarrow \mathbb{N}$ . We write  $\Sigma_r = (\Sigma, ar)$ . The idea is that each symbol comes pre-equipped with a number which indicates how many children it has in trees. This is reasonable provided a node's label determines how many children it can have.

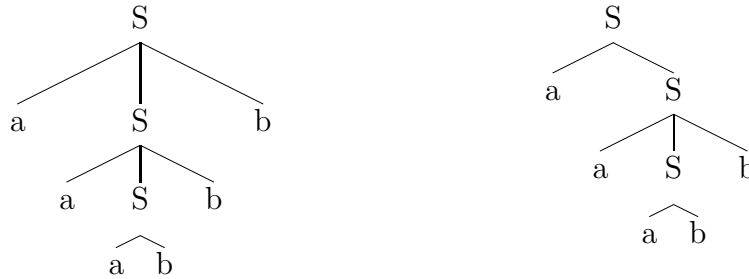
Strictly speaking, a ranked alphabet is not a necessary feature of tree automata. There are two substantive reasons to adopt it. First, using it helps ensure that the transition function is finite. (So it can accomplish the same thing as our requirement that the pre-image of  $\delta$  be finite.) Second, it helps ensure our transition function is total; that is, defined for every element of the alphabet and the possible states of its children.

### 4.1.4 Examples

**Example 7.** Let  $A = (Q, \Sigma, F, \delta)$  with its parts defined as follows.

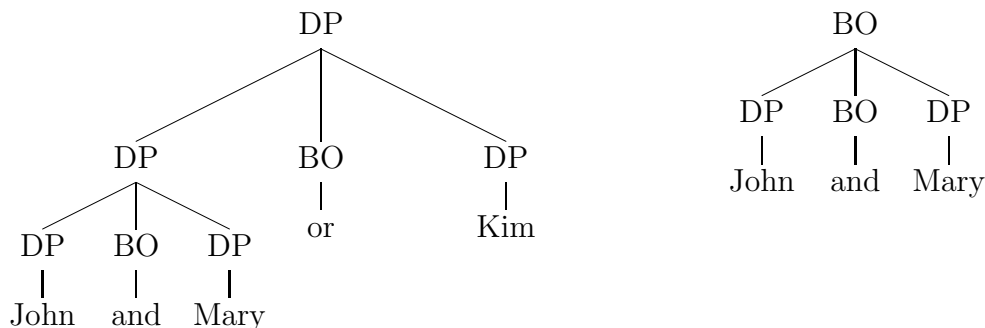
- $Q = \{q_a, q_b, q_S\}$
- $\Sigma = \{a, b, S\}$
- $F = \{q_S\}$
- $\delta(\lambda, a) = q_a$
- $\delta(\lambda, b) = q_b$
- $\delta(q_a q_b, S) = q_S$
- $\delta(q_a q_S q_b, S) = q_S$

Let us see how the acceptor  $A$  processes the two trees below as inputs.



**Example 8.** Let  $A = (Q, \Sigma, F, \delta)$  with its parts defined as follows.

- $Q = \{q_{DP}, q_{BO}\}$
- $\Sigma = \{\text{and, or, Kim, John, Mary, DP, BO}\}$
- $F = \{q_{DP}\}$
- $\delta(q_{DP}, DP) = q_{DP}$
- $\delta(q_{BO}, BO) = q_{BO}$
- $\delta(q_{DP} q_{BO} q_{DP}, DP) = q_{DP}$
- $\delta(\lambda, \text{Kim}) = q_{DP}$
- $\delta(\lambda, \text{John}) = q_{DP}$
- $\delta(\lambda, \text{Mary}) = q_{DP}$
- $\delta(\lambda, \text{and}) = q_{BO}$
- $\delta(\lambda, \text{or}) = q_{BO}$



### 4.1.5 Observations

- For every symbol  $a \in \Sigma$  which can be leaf in a tree, you will need to define a transition  $\delta(\lambda, a)$ .
- For every symbol  $a \in \Sigma$  which can have  $n$  children, you will need to define a transition  $\delta(q_1 \cdots q_n, a)$ .

### 4.1.6 Connection to Context-Free Languages

Recognizable treesets are closely related to the derivation trees of context-free languages.

#### Theorem 8.

- Let  $G$  be a context-free word grammar, then the set of derivation trees of  $L(G)$  is a recognizable tree language.
- Let  $L$  be a recognizable tree language then  $\text{Yield}(L)$  is a context-free word language.
- There exists a recognizable tree language not equal to the set of derivation trees of any context-free language. Thus the class of derivation treesets of context-free word languages is a proper subset of the class of recognizable treesets.

## 4.2 Deterministic Top-down Finite-state Tree Acceptors

### 4.2.1 Orientation

This section is about deterministic top-down finite-state tree acceptors. The term *finite-state* means that the memory is bounded by a constant, no matter the size of the input to the machine. The term *deterministic* means there is single course of action the machine follows to compute its output. The term *acceptor* means this machine solves *membership problem*: given a set of objects  $X$  and input object  $x$ , does  $x$  belong to  $X$ ? The term *tree* means we are considering the membership problem over *treesets*. The term *top-down* means that for each node  $a$  in a tree, the computation solves the problem by assigning a state to the parent

of  $a$  before assigning a state to  $a$  itself. This contrasts with *bottom-up* machines which assign states to the children of  $a$  first and then  $a$ . Visually, these terms make sense provided the root of the tree is at the top and branches of the tree move downward.

*Acceptor* is synonymous with *recognizer*. *Treeset* is synonymous with *tree language*.

A definitive reference for finite-state automata for trees is freely available online. It is “Tree Automata Techniques and Applications” (TATA) (Comon *et al.*, 2007). The presentation here differs from the one there, as mentioned below.

## 4.2.2 Definition

**Definition 21** (DTFTA). *A Deterministic Top-down Finite-state Acceptor (DTFTA) is a tuple  $(Q, \Sigma_r, F, \delta)$  where*

- $Q$  is a finite set of states;
- $q_0$  is a initial state;
- $\Sigma$  is a finite alphabet;
- $\delta : Q \times \Sigma \times \mathbb{N} \rightarrow Q^*$  is the transition function. Note the pre-image of  $\delta$  is necessarily finite.

The transition function takes a state, a letter, and a number  $n$  and returns a string of states. The idea is that the length of this output string should be  $n$ . Basically, when moving top-down, the states of the child sub-trees depend on these three things: the state of the parent, the label of the parent, and the number of children the parent has.

We use the transition function  $\delta$  to define a new function  $\delta^* : Q \times \Sigma^T \rightarrow Q^*$  as follows.

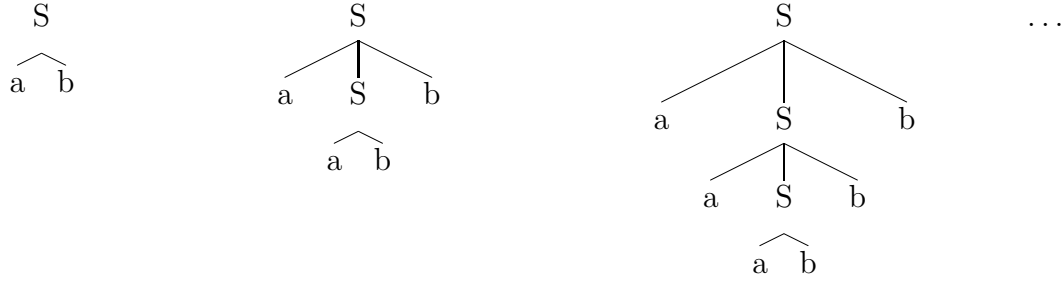
$$\begin{aligned} \delta^*(q, a[\lambda]) &= \delta(q, a, 0) \\ \delta^*(q, a[t_1 \cdots t_n]) &= \delta^*(q_1, t_1) \cdots \delta^*(q_n, t_n) \text{ where } \delta(q, a, n) = q_1 \cdots q_n \end{aligned} \quad (4.2)$$

As before, there are some important consequences to the formulation of  $\delta^*$ . One is that  $\delta^*$  is undefined on tree  $a[t_1 \cdots t_n]$  if transition  $\delta(q, a, n)$  does not return a string from  $Q^*$  of length  $n$ . (Also, I am abusing notation since  $\delta^*$  is strictly speaking not the transitive closure of  $\delta$ .)

**Definition 22** (Treeset of a DTFTA). *Consider some DTFTA  $A = (Q, \Sigma, F, \delta)$  and tree  $t \in \Sigma^T$ . If  $\delta^*(q_0, t)$  is defined and equals  $\lambda$  then we say  $A$  accepts/recognizes  $t$ . Otherwise  $A$  rejects  $t$ . Formally, the treeset recognized by  $A$  is  $L(A) = \{t \in \Sigma^T \mid \delta^*(t) = \lambda\}$ .*

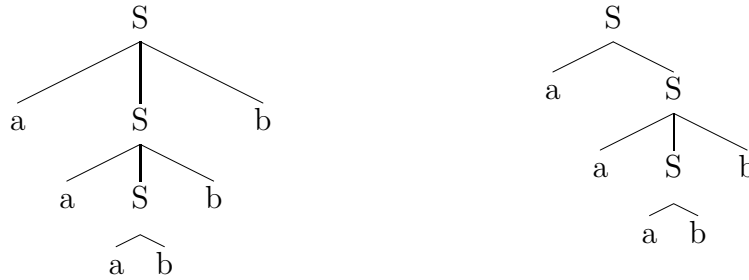
The use of the ‘L’ denotes “Language” as treesets are traditionally referred to as *formal tree languages*.

**Example 9.** Recall the example from last week which generates trees like



- $Q = \{q_a, q_b, q_S\}$
- $\Sigma = \{a, b, S\}$
- $q_0 = q_S$
- $\delta(q_a, a, 0) = \lambda$
- $\delta(q_b, b, 0) = \lambda$
- $\delta(q_S, S, 3) = q_a q_S q_b$
- $\delta(q_S, S, 2) = q_a q_b$

Let us see how the acceptor  $A$  processes the two trees below as inputs.



**Theorem 9.** *Every treeset recognizable by a DTFTA is recognizable, but there are recognizable treesets which cannot be recognized by a DTFTA.*

The following example helps show why this is the case. Consider the treeset  $T$  containing only the two trees shown below.



This is a recogniiable treeset because the DBFTA below recognizes exactly these two trees and no others.

- $Q = \{q, q_S\}$
- $\Sigma = \{a, b, S\}$
- $q_0 = q_S$
- $\delta(\lambda, a) = q_a$
- $\delta(\lambda, b) = q_b$
- $\delta(q_a q_b, S) = q_S$
- $\delta(q_b q_a, S) = q_S$

Notice that this DBFTA fails on these two trees.





A DTFTA cannot recognize the trees in  $T$  without also recognizing the trees shown immediately above. This is because moving top down there can only be one value for  $\delta(q_S, S, 2)$ . Suppose it equals  $q_1q_2$ . To recognize the first tree, we would also have to make sure that  $\delta(q_1, a, 0)$  and  $\delta(q_2, b, 0)$  are defined. Similarly, to recognize the second tree, we would have to make sure that  $\delta(q_1, b, 0)$  and  $\delta(q_2, a, 0)$  are defined. But it follows then that the aforementioned trees above are also recognized by this DTFTA. For instance the tree with two  $a$  leaves is recognized because both  $\delta(q_1, a, 0)$  and  $\delta(q_2, a, 0)$  are defined. Thus no DTFTA recognizes  $T$ .

### 4.2.3 Observations

- For every symbol  $a \in \Sigma$  which can be leaf in a tree, you will need to define a transition  $\delta(q, a, 0) = \lambda$ .
- For every symbol  $a \in \Sigma$  which can have  $n$  children, you will need to define a transition  $\delta(q, a, n) = q_1 \cdots q_n$ .

## 4.3 Properties of recognizable tree languages

**Theorem 10** (Closure under Boolean operations). *The class of recognizable tree languages is closed under union, under complementation, and under intersection.*

The proofs of these cases are very similar to the ones for finite-state acceptors over strings. For every DBFTA  $A$  recognizing a treeset  $T$ , it can be made *complete* by adding a sink state and transitions to it. Then product constructions can be used to establish closure under union and intersection. Closure under complement is established the same as before too: everything is the same except the final states are now the non-final states of  $A$ .

**Theorem 11** (Minimal, deterministic, canonical form). *For every recognizable tree language  $T$ , there is a unique, smallest DBFTA  $A$  which recognizes  $T$ . That is, if DBFTA  $A'$  also recognizes  $T$  then there at least as many states in  $A'$  as there are in  $A$ .*

## 4.4 Connection to Context-Free Languages

Context Free Grammars (CFGs) are studied in detail in a number of textbooks including Harrison (1978); Davis and Weyuker (1983); Hopcroft *et al.* (2001) and Sipser (1997).

**Definition 23.** *A rewrite grammar is a tuple  $\langle T, N, S, \mathcal{R} \rangle$  where*

- $\mathcal{T}$  is a nonempty finite alphabet of symbols. These symbols are also called the terminal symbols, and we usually write them with lowercase letters like  $a, b, c, \dots$
- $\mathcal{N}$  is a nonempty finite set of non-terminal symbols, which are distinct from elements of  $\mathcal{T}$ . These symbols are also called category symbols, and we usually write them with uppercase letters like  $A, B, C, \dots$

- $S$  is the start category, which is an element of  $\mathcal{N}$ .
- A finite set of production rules  $\mathcal{R}$ . A production rule has the form  $\alpha \rightarrow \beta$  where  $\alpha, \beta$  belong to  $(\mathcal{T} \cup \mathcal{N})^*$ . In other words,  $\alpha$  and  $\beta$  are strings of non-terminal and terminal symbols. While  $\beta$  may be the empty string we require that  $\alpha$  include at least one symbol.

Rewrite grammars are also called *phrase structure grammars*.

**Definition 24.** A CFG is a rewrite grammar with the following properties.

- For all rules  $\alpha \rightarrow \beta$ ,  $\alpha$  is an element of  $\mathcal{N}$ . So the left-hand-side of each rule is a single non-terminal.
- For all rules,  $\alpha \rightarrow \beta$ ,  $\beta$  is an element of  $(\mathcal{N} \cup \mathcal{T})^*$ . So the right-hand-side of each rule is a sequence of non-terminal symbols or a single terminal symbol.

**Example 10.** Define a CFG  $G_{NE}$  as follows. Let  $\mathcal{N} = \{A, B, S\}$  and  $\mathcal{T} = \{a, b\}$ . Let  $\mathcal{R}$  include the rules  $S \rightarrow a S b$  and  $S \rightarrow a b$ .

Next we define the languages of the rewrite grammars in addition to the derivation treeset of context free grammars.

The language of a rewrite grammar is defined recursively below.

**Definition 25.** The (partial) derivations of a rewrite grammar  $G = \langle \mathcal{T}, \mathcal{N}, S, \mathcal{R} \rangle$  is written  $D(G)$  and is defined recursively as follows.

1. The base case:  $S$  belongs to  $D(G)$ .
2. The recursive case: For all  $\alpha \rightarrow \beta \in \mathcal{R}$  and for all  $\gamma_1, \gamma_2 \in (\mathcal{T} \cup \mathcal{N})^*$ , if  $\gamma_1 \alpha \gamma_2 \in D(G)$  then  $\gamma_1 \beta \gamma_2 \in D(G)$ .
3. Nothing else is in  $D(G)$ .

Then the language of the grammar  $L(G) = \{w \in \mathcal{T}^* \mid w \in D(G)\}$ .

**Exercise 16.** Using the definition above, explain why  $aaabbb$  belongs to  $L(G_{NE})$ .

The derivation treeset of a context free grammar is defined recursively below. The *derivation treeset* of a CFG  $G = \langle \mathcal{T}, \mathcal{N}, S, \mathcal{R} \rangle$  is written  $D_T(G)$ .

It is defined as all and only those trees  $t$  such that

1.  $\text{yield}(t) \in L(G)$  and
2. for all non-leaf nodes  $a[a_1[ts_1]a_2[ts_2] \cdots a_n[ts_n]]$  in  $t$ , the rule  $a \rightarrow a_1 a_2 \cdots a_n$  belongs to  $\mathcal{R}$ .

Recognizable treesets are closely related to the derivation trees of context-free languages.

**Theorem 12.**

- Let  $G$  be a context-free word grammar, then the set of derivation trees  $D_T(G)$  is a recognizable tree language.

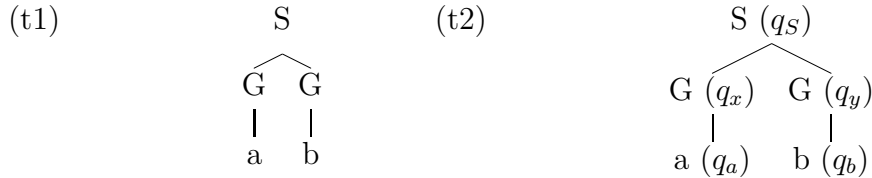
- *There exists a recognizable tree language not equal to the set of derivation trees of any context-free language. Thus the class of derivation treesets of context-free word languages is a proper subset of the class of recognizable treesets.*
- *Let  $L$  be a recognizable tree language then  $\mathit{yield}(L)$  is a context-free word language.*

Each of these has a straightforward explanation.

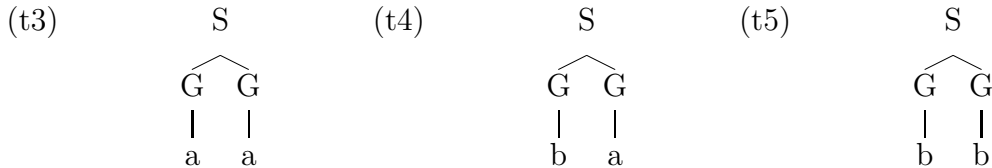
For (1), the recognizable tree language which recognizes  $D_T(G)$  for a CFG  $G$  can be constructed based on the rules of  $G$ . For each symbol  $a$  in  $N$ , the DBFTA should include  $\delta(\lambda, a) = q_a$ . And, for each rule  $A \rightarrow B_1 \cdots B_n$  in  $R$ , the DBFTA should include  $\delta(q_{B_1} \cdots q_{B_n}, A) = q_A$ . That's it!

For (2), consider the DBFTA  $A$  shown below. The claim is that there is no CFG whose derivation language is exactly this recognizable treeset. The only tree in  $L(A)$  is (t1), which is shown below  $A$  at left. Tree (t2) shows (t1) with the states  $A$  assigns to its subtrees.

$$\begin{array}{ll}
 (A) \quad Q & = \{q_S, q_a, q_b, q_x, q_y\} & \delta(\lambda, a) & = q_a \\
 & \Sigma = \{S, G, a, b\} & \delta(\lambda, b) & = q_b \\
 & F = \{q_S\} & \delta(q_a, G) & = q_x \\
 & \delta(q_x q_y, S) & = q_S & \delta(q_b, G) & = q_y
 \end{array}$$



That no CFG can recognize this treeset follows from the fact that such any CFG  $G$  which includes the tree above will need to have the following rules:  $S \rightarrow GG, G \rightarrow a, G \rightarrow b$ . But then this  $G$  will not only generate (t1) above but also the derivation trees shown below.



Thus  $L(A) \neq D_T(G)$ .

This example shows that the states of the DBFTA are more abstract than the labels on the nodes. The reason recognizable tree languages are more expressive than the derivation treesets of CFGs follows from this. The DBFTA uses states  $q_x$  and  $q_y$  to distinguish the subtrees bearing the label  $G$ . But the CFG cannot distinguish these trees in this way.

For (3), observe that we can write a CFG that generates the same stringset as the one above. For instance, we could write a CFG with the rule  $S \rightarrow ab$ .

More generally, though, we can always write a CFG that puts the state information into the nodes themselves. The trees in the derivation treeset for this CFG would be “structurally the same” as the trees in the recognizable treeset, but the labels on the nodes would be

different. So they are not the same trees. In the example above for instance we can write a CFG with rules  $S \rightarrow G_x G_y$ ,  $G_x \rightarrow a$ ,  $G_x \rightarrow b$ . There is only one tree in this CFG's derivation treeset shown below.



Importantly, (t6) is not the same as (t1). The inner nodes are labeled differently! So they are different trees. However, they only differ with respect to how the inner nodes are labeled, so it follows that the stringsets obtained by taking the **yield** of these trees are the same. This is the kind of argument used to show that the yield of any recognizable treeset is a context-free language.

# Chapter 5

## Tree Transducers

### 5.1 Deterministic Bottom-up Finite-state Tree Transducers

#### 5.1.1 Orientation

This section is about deterministic bottom-up finite-state tree transducers. The term *finite-state* means that the amount of memory needed in the course of computation is independent of the size of the input. The term *deterministic* means there is single course of action the machine follows to compute its output. The term *transducer* means this machine solves *transformation problem*: given an input object  $x$ , what object  $y$  is  $x$  transformed into? The term *tree* means we are considering the transformation problem from trees to trees. The term *bottom-up* means that for each node  $a$  in a tree, the computation transforms the children of a node before transforming the node. This contrasts with *top-down* transducers which transform the children after transforming their parent. Visually, these terms make sense provided the root of the tree is at the top and branches of the tree move downward.

A definitive reference for finite-state automata for trees is freely available online. It is “Tree Automata Techniques and Applications” (TATA) (Comon *et al.*, 2007). The presentation here differs from the one there, as mentioned below.

#### 5.1.2 Definitions

Recall the definition of trees with a finite alphabet  $\Sigma$  and the symbols  $[ ]$  not in  $\Sigma$ . All such trees belonged to the treeset  $\Sigma^T$ . In addition to this, we will need to define a new kind of tree which has *variables* in the leaves. I will call these trees *Variably-Leafed*. We assume a countable set of variables  $X$  containing variables  $x_1, x_2, \dots$ .

**Definition 26** (Variably-Leafed Trees).

**Base Cases ( $\Sigma$ ):** For each  $a \in \Sigma$ ,  $a[ ]$  is a tree.

**Base Cases ( $X$ ):** For each  $x \in X$ ,  $x[ ]$  is a tree.

**Inductive Case:** If  $a \in \Sigma$  and  $t_1 t_2 \dots t_n$  is a string of trees of length  $n$  then  $a[t_1 t_2 \dots t_n]$  is a tree.

Let  $\Sigma^T[X]$  denote the set of all variably-leafed trees of finite size using  $\Sigma$  and  $X$ .

Note that  $\Sigma^T \subsetneq \Sigma^T[X]$ . In the tree transducers we write below, the variably-leafed trees will play a role in the omega function as well as the the intermediate stages of the transformation.

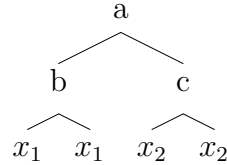
With this definition in place, we can define our first tree transducer.

**Definition 27** (DBFTA). A Deterministic Bottom-up Finite-state Acceptor (DBFTT) is a tuple  $(Q, \Sigma_r, F, \delta)$  where

- $Q$  is a finite set of states;
- $\Sigma$  is a finite alphabet;
- $F \subseteq Q$  is a set of accepting (final) states; and
- $\delta : Q^* \times \Sigma \rightarrow Q$  is the transition function. The pre-image of  $\delta$  must be finite. This means we can write it down—for example, as a list.
- $\Omega$  is a function with domain  $Q^* \times \Sigma$  and co-domain  $\Sigma^T[X]$ . Its pre-image must also be finite.

Generally, the pre-images of  $\delta$  and  $\Omega$  should coincide.

**Example 11.** Let  $M$  be a DBFTT and suppose  $\Omega(q_1, q_2, a) = a[ b[x_1 \ x_1] \ c[x_2 \ x_2] ]$ . So this is the variably-leafed tree shown below.

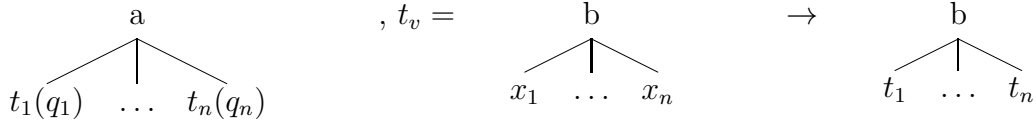


Further suppose  $\Omega(\lambda, c) = c[ ]$  and  $\Omega(\lambda, d) = d[ ]$ . The idea is that  $M$  will transform the tree shown below at left into the tree shown below at right.

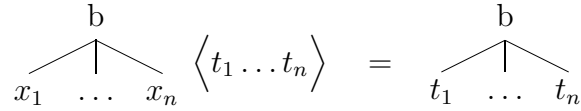


Intuitively, this is because each variable  $x_i$  in the variably leafed tree will be replaced by the  $i$ th child of the root node of the tree under consideration.

Here is the general schema. Given  $\Omega(q_1 q_2 \dots q_n, a) = t_v$  and a tree  $a[t_1 t_2 \dots t_n]$  with states  $q_1 q_2 \dots q_n$ , respectively, then the output tree will be the one obtained by replacing each  $x_i$  with  $t_i$  in  $t_v$ . A schematic of this is shown below.



We now formalize the above ideas. As before, we extend the transition function  $\delta$  to  $\delta^* : \Sigma^T \rightarrow Q$ . If  $t_1 \dots t_n$  is a list of trees and  $t_v \in \Sigma^T[X]$  is a variably leafed tree with variables  $x_1, \dots, x_n$  then let  $t_v \langle t_1 \dots t_n \rangle = t \in \Sigma^T$  obtained by replacing each variable  $x_i$  in  $t_v$  with  $t_i$ . Here is another visualization with this notation.



We also define a new function “process”  $\pi : \Sigma^T \rightarrow Q \times \Sigma^T$  which will process the tree and produce its output. It is defined as follows.

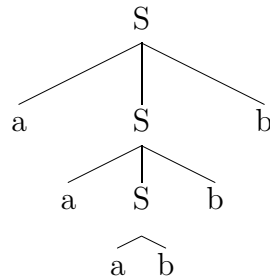
$$\begin{aligned}
 \pi(a[ \ ] ) &= (\delta(\lambda, a), \Omega(\lambda, a)) \\
 \pi(a[t_1 \dots t_n]) &= (q, t) \\
 &\text{where } q = \delta(q_1 \dots q_n, a) \\
 &\text{and } t = \Omega(q_1 \dots q_n, a) \langle s_1 \dots s_n \rangle \\
 &\text{and } (q_1, s_1) \dots (q_n, s_n) = \pi(t_1) \dots \pi(t_n)
 \end{aligned} \tag{5.1}$$

**Definition 28** (Tree-to-tree function of a DBFTT). *The function defined by the transducer  $M$  is  $\{(t, s) \mid t, s \in \Sigma^T, \pi(t) = (q, s), q \in F\}$ . If  $(t, s)$  belongs to this set, we say  $M$  transduces  $t$  to  $s$  and write  $M(t) = s$ .*

**Example 12.** Consider the transducer  $M$  constructed as follows.

- $Q = \{q_a, q_b, q_S\}$
- $\Sigma = \{a, b, S\}$
- $F = \{q_S\}$
- $\delta(\lambda, a) = q_a$
- $\delta(\lambda, b) = q_b$
- $\delta(q_a q_b, S) = q_S$
- $\delta(q_a q_S q_b, S) = q_S$
- $\Omega(\lambda, a) = a[ \ ]$
- $\Omega(\lambda, b) = b[ \ ]$
- $\Omega(q_a q_b, S) = S[x_2 x_1]$
- $\Omega(q_a q_S q_b, S) = S[x_3 x_2 x_1]$

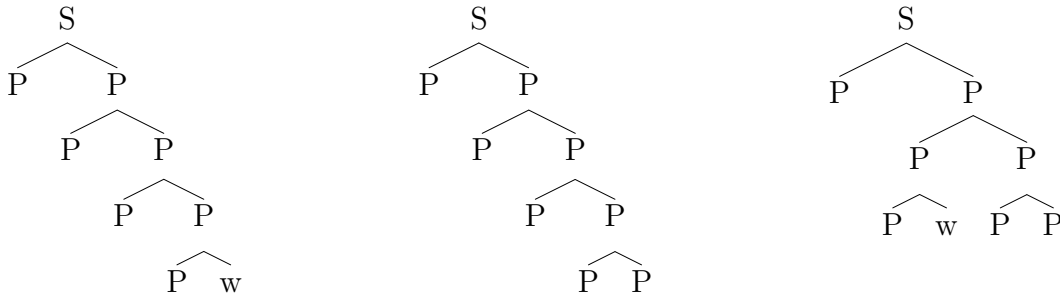
Let us work out what  $M$  transforms the tree below into.



**Example 13.** Consider the transducer  $M$  constructed as follows. We let  $Q = \{q_w, q_p, q_s\}$ ,  $\Sigma = \{w, P, S\}$ , and  $F = \{q_s\}$ . The transition and output functions are given below.

- $\delta(\lambda, P) = q_p$
- $\delta(\lambda, w) = q_w$
- $\delta(q_p q_p, P) = q_p$
- $\delta(q_w q_p, P) = q_w$
- $\delta(q_p q_w, P) = q_w$
- $\delta(q_p q_w, S) = q_s$
- $\delta(q_w q_p, S) = q_s$
- $\delta(q_p q_p, S) = q_s$
- $\Omega(\lambda, P) = P[ \ ]$
- $\Omega(\lambda, w) = w[ \ ]$
- $\Omega(q_p q_p, P) = P[x_1 x_2]$
- $\Omega(q_w q_p, P) = P[x_1 x_2]$
- $\Omega(q_p q_w, P) = P[x_1 x_2]$
- $\Omega(q_p q_w, S) = S[ w[ \ ] S[x_1 x_2] ]$
- $\Omega(q_w q_p, S) = S[ w[ \ ] S[x_1 x_2] ]$
- $\Omega(q_p q_p, S) = S[x_1 x_2]$

Let us work out how  $M$  transforms the trees below.



## 5.2 Deterministic Top-down Finite-state Tree Transducers

### 5.2.1 Orientation

This section is about deterministic bottom-up finite-state tree transducers. The term *finite-state* means that the amount of memory needed in the course of computation is independent of the size of the input. The term *deterministic* means there is single course of action the machine follows to compute its output. The term *transducer* means this machine solves *transformation problem*: given an input object  $x$ , what object  $y$  is  $x$  transformed into? The term *tree* means we are considering the transformation problem from trees to trees. The term *top-down* means that for each node  $a$  in a tree, the computation transforms the node before transforming its children. This contrasts with *bottom-up* transducers which transform the children before transforming their parent. Visually, these terms make sense provided the root of the tree is at the top and branches of the tree move downward.

A definitive reference for finite-state automata for trees is freely available online. It is “Tree Automata Techniques and Applications” (TATA) (Comon *et al.*, 2007). The presentation here differs from the one there, as mentioned below.



## 5.2.2 Definitions

As before, we use variably leafed trees  $\Sigma^T[X]$ .

**Definition 29** (DTFTT). *A Deterministic Top-down Finite-state Acceptor (DTFTT) is a tuple  $(Q, \Sigma, q_0, \delta)$  where*

- $Q$  is a finite set of states;
- $\Sigma$  is a finite alphabet;
- $q_0 \in Q$  is the initial state; and
- $\delta : Q \times \Sigma \times \mathbb{N} \rightarrow Q^*$  is the transition function.
- $\Omega$  is a function with domain  $Q \times \Sigma \times \mathbb{N}$  and co-domain  $\Sigma^T[X]$ .

Generally, the pre-images of  $\delta$  and  $\Omega$  should coincide.

We also define a new function “process”  $\pi : Q \times \Sigma^T \rightarrow \Sigma^T$  which will process the tree and produce its output. It is defined as follows.

$$\begin{aligned} \pi(q, a[ ]) &= \Omega(q, a, 0) \\ \pi(q, a[t_1 \cdots t_n]) &= \Omega(q, a, n) \langle \pi(q_1, t_1) \cdots \pi(q_n, t_n) \rangle \\ &\quad \text{where } q_1 \cdots q_n = \delta(q, a, n) \end{aligned} \tag{5.2}$$

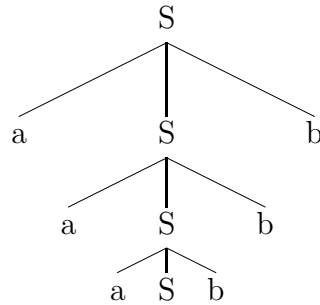
Intuitively,  $\Omega$  transforms the root node into a variably leafed tree. The variables are replaced with the children of the root node. These children are also trees with states assigned by  $\delta$ . Then  $\pi$  transforms each tree-child as well.

**Definition 30** (Tree-to-tree function of a DTFTT). *The function defined by the transducer  $M$  is  $\{(t, s) \mid t, s \in \Sigma^T, \pi(q_0, t) = s\}$ . If  $(t, s)$  belongs to this set, we say  $M$  transduces  $t$  to  $s$  and write  $M(t) = s$ .*

**Example 14.** Consider the transducer  $M$  constructed as follows.

- $Q = \{q, q_S\}$
- $\Sigma = \{a, b, S\}$
- $q_0 = q_S$
- $\delta(q, a, 0) = \lambda$
- $\delta(q, b, 0) = \lambda$
- $\delta(q_S, S, 3) = qqSq$
- $\delta(q_S, S, 2) = qq$
- $\Omega(q, a, 0) = a[ ]$
- $\Omega(q, b, 0) = b[ ]$
- $\Omega(q, S, 3) = S[x_3x_2x_1]$
- $\Omega(q, S, 2) = S[x_2x_1]$

Let see how  $M$  transforms the tree below.



**Exercise 17.** Recall the “wh-movement” example from before. Explain why this transformation *cannot* be computed by a deterministic top-down tree transducer.

### 5.3 Theorems about Deterministic Tree Transducers

**Theorem 13** (composition closure). *The class of deterministic bottom-up transductions is closed under composition, but the class of top-down deterministic transductions is not.*

**Theorem 14** (Incomparable). *The class of deterministic bottom-up transductions is incomparable with the the class of top-down deterministic transductions.*

This theorem is based on the same kind of examples which separated the left and right sequential functions. Let relations  $U = (f^na, f^na) \mid n \in \mathbb{N} \cup (f^nb, g^nb) \mid n \in \mathbb{N}$  and  $D = (ff^na, ff^na) \mid n \in \mathbb{N} \cup (gf^na, gf^nb) \mid n \in \mathbb{N}$ .  $U$  is recognized by a DBFTT but not any DTFTT and  $D$  is recognized by a DTFTT but not any DBFTT.

# Chapter 6

## Nondeterminism

### 6.1 Non-Determinism

So far all of the finite-state automata we have considered have been deterministic. Informally, a computation is *deterministic* provided that as the computation unfolds for any input, at each step in the computation, there is at most one “next step.” On the other hand, for *non-deterministic* computations, there may be multiple “next steps” in the course of the computation. Multiple output forms would seem to imply that at some point in the computation there need be multiple “next steps.”

In the case of the automata we have looked at so far, there was at most one initial state. Also, the delta function had at most one output. Consequently, there was at most one path through any given automata for a particular input string or tree.

In this section, we study what happens if we add non-determinism into the picture. In other words, we will effectively change the delta *function* to a *relation* and allow more than one initial state.

As it turns out, in terms of the class of recognizable stringsets and treesets, the addition of non-determinism has little consequence. However, for the classes of string and tree transductions, the effects of non-determinism are significant.

#### 6.1.1 Non-deterministic string acceptors

**Definition 31.** A non-deterministic finite-state acceptor (NFA) is a tuple  $(Q, \Sigma, I, F, \delta)$  where

- $Q$  is a finite set of states;
- $\Sigma$  is a finite set of symbols (the alphabet);
- $I \subseteq Q$  is a set of initial states;
- $F \subseteq Q$  is a set of accepting (final) states; and
- $\delta$  is a function with domain  $Q \times \Sigma \cup \{\epsilon\}$  and co-domain  $\wp(Q)$ . It is called the transition function.

For any set  $A$ ,  $\wp(A)$  denotes the powerset of  $A$ , which is the set of all subsets of  $A$ . The symbol  $\epsilon$  denotes a “free” change of state; that is the state of the system can change without any input being consumed.

The *epsilon closure* of a set of states  $S \subseteq Q$  is the smallest subset  $S'$  of  $Q$  satisfying these properties:

1.  $S \in S'$
2. For all  $q, r \in Q$ : if  $q \in S'$  and  $r \in \delta(q, \epsilon)$  then  $r \in S'$ .

The epsilon closure of  $Q$  is denoted  $C_\epsilon(Q)$ .

We extend the domain of the transition function to  $\wp(Q) \times \Sigma^*$  as follows.

First we define  $\delta' : \wp(Q) \times \Sigma \rightarrow \wp(Q)$  as follows:  $\delta(Q, a) = \bigcup_{q \in Q} \delta(q, a)$

$$\begin{aligned} \delta^*(Q, \lambda) &= Q \\ \delta^*(Q, aw) &= \delta^*(C_\epsilon(Q'), w) \text{ where } Q' = \delta'(C_\epsilon(Q), a) \end{aligned} \tag{6.1}$$

Consider some DFA  $A = (Q, \Sigma, I, F, \delta)$  and string  $w \in \Sigma^*$ . If  $\delta^*(I, w) \cap F \neq \emptyset$  then we say  $A$  *accepts/recognizes/generates*  $w$ . Otherwise  $A$  *rejects*  $w$ .

**Definition 32.** *The stringset recognized by  $A$  is  $L(A) = \{w \in \Sigma^* \mid \delta^*(I, w) \cap F \neq \emptyset\}$ .*

**Exercise 18.**

1. Non-determinism makes expressing some stringsets easier because it can be done with fewer states.
  - (a) Write an acceptor for the set of strings where the last letter must be a consonant.
  - (b) Write an acceptor for the set of strings where the second-to-last letter must be a consonant.
  - (c) Write an acceptor for the set of strings where the third-to-last letter must be a consonant.
2. Use the plurality of initial states to show that if  $R$  and  $S$  are stringsets each recognized by some NFA that the union  $R \cup S$  is also recognized by a NFA.
3. Use epsilon transitions to show that if  $R$  and  $S$  are stringsets each recognized by some NFA that the concatenation  $RS$  is also recognized by a NFA.

**Theorem 15.** *The class of stringsets recognizable by the NFA are exactly the regular stringsets.*

This theorem states that there is no gain in expressivity by introducing non-determinism into the models. However, as we have seen there can be gains practically in terms of how we write the models.

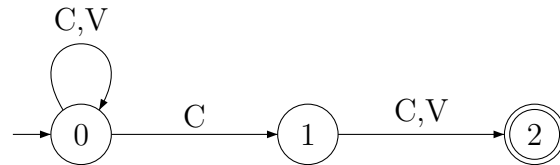
The proof of the above theorem shows how, for any NFA with state set  $Q$ , to construct an equivalent DFA. The construction is often called *the subset construction* because the states in the DFA will correspond to subsets of  $Q$ .

Here is the basic idea (ignoring epsilon transitions.) Consider any NFA  $N = (Q_N, \Sigma, I_N, F_N, \delta_N)$ . Then we can construct an equivalent DFA  $D = (Q_D, \Sigma, I_D, F_D, \delta_D)$  as follows.

1.  $Q_D = \wp(Q_N)$
2.  $I_D = I_N$
3.  $F_D = \{S \mid S \cap F_N \neq \emptyset\}$
4.  $\delta_D(S, a) = \bigcup_{q \in S} \delta_N(q, a)$

Essentially, sets of states in the NFA correspond to distinct states in the equivalent DFA. (If the NFA includes epsilon transitions, it will be important to take into account the epsilon closure of  $S$  and the states reachable from  $S$  on  $a$  in the calculation  $\delta_D(S, a)$ .)

Here is an example. Consider the NFA below.

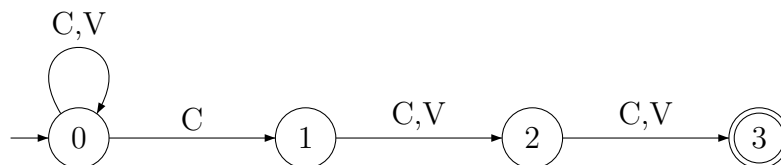


Here is the full transition table according to the subset construction. Final states are marked with the \* and the initial state with  $\rightarrow$ .

$Q_D$	C	V
$\emptyset$	$\emptyset$	$\emptyset$
$\rightarrow\{0\}$	$\{0,1\}$	$\{0\}$
$\{1\}$	$\{2\}$	$\emptyset$
$\{2\}$	$\emptyset$	$\emptyset$
$\{0,1\}$	$\{0,1,2\}$	$\{0,2\}$
$\{0,2\}$	$\{0,1\}$	$\{0\}$
$\{1,2\}$	$\{2\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$	$\{0,2\}$

**Exercise 19.** Let's draw what this looks like. Which states are useless? If the useless states are removed, is the resulting machine complete?

**Exercise 20.** Write the DFA equivalent to the NFA shown below.



### 6.1.2 Tree acceptors

**Theorem 16.** *The class of treesets recognized by non-deterministic bottom-up acceptors is exactly the recognizable treesets (so the class of treesets recognized by deterministic bottom-up acceptors).*

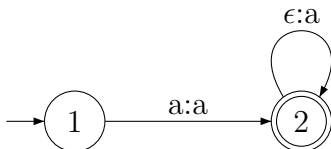
**Theorem 17.** *The class of treesets recognized by non-deterministic top-down acceptors is also exactly the recognizable treesets.*

### 6.1.3 Non-deterministic string transducers

**Theorem 18.** *The left projection (domain) of a regular relation is a regular stringset. The right projection (co-domain) of a regular relation is a regular stringset.*

**Theorem 19.** *Left and right sequential functions are proper subclasses of the relations recognized by non-deterministic string transducers.*

One reason why this is true is that epsilon transitions permit non-deterministic transducers to recognize *infinite relations* not functions. For example, the non-deterministic acceptor below relates input  $a$  to every element in  $\{a, aa, aaa, \dots\}$ .



This is not possible for sequential transducers with the  $\Sigma^*$  monoid or the finite language monoid. However, it should be possible with the “Regular language” monoid where the outputs of the transitions are actually NFA that get “concatenated.”

### 6.1.4 Tree transducers

Recall that a tree transducer is *linear* when the variable trees in the omega function include at most one of each variable. (In other words, trees are not copied.)

**Theorem 20.** *The domain of a non-deterministic tree transducer is a recognizable tree language. The image of a recognizable tree language by a linear tree transducer is recognizable.*

**Theorem 21.** *Generally, the class of tree relations recognized by non-deterministic top-down tree transducers and non-deterministic bottom-up tree transducers are incomparable.*

**Theorem 22.** *However, any linear top-down tree transducer is equivalent to a linear bottom-up tree transducer.*

### 6.1.5 Summary

For acceptors recognizing sets of strings or trees, non-determinism does NOT entail an increase in expressivity as the subset construction shows. However, non-deterministic computations can require exponentially fewer states.

For transducers, the current understanding is that non-determinism DOES lead to more expressive constructions. The ability of transducers to recognize infinite relations is usually taken to be the key example. However, there are examples that do not require infinite relations as discussed in Heinz and Lai (2013).

There is much more to cover here more generally, see Hopcroft *et al.* (2001) and Comon *et al.* (2007)





# Chapter 7

## Conclusion

### 7.1 Conclusion

In this class, we learned about finite-state automata and how they can be used to express linguistic generalizations.

We studied two kinds of linguistic generalizations: representational and transformational. The former asks whether a particular representation is well-formed or not. The latter changes one representation into another. From a computational perspective, the first question corresponds to a *membership problem* and the second one to what we called a *transformation problem*.

Broadly speaking, automata are machines that solve particular classes of these problems. *Recognizers*, also called *acceptors*, solve membership problems. *Transducers* solve transformation problems.

**When the automata are finite-state, it means that the memory they require is bounded by a constant, no matter the size of the input.**

We studied four major classes of finite-state automata: recognizers and transducers for strings and trees. We implemented them in the programming language Haskell, a fully-typed, functional programming language with lazy evaluation. The programs were short and concise, and the code looked a lot like the math.

In the remainder of this conclusion, I want to emphasize aspects of automata that we were unable to get into in class.

#### 7.1.1 Two-way deterministic automata

The automata we studied are called one-way automata. This means they processed the data structures in one direction. For instance, for strings the automata processed them left-to-right. For trees, it was either top-down or bottom-up. However, people have also studied two-way automata. These automata are allowed to re-read parts of the input string.

A useful analogy is to consider the automata as a scanning device with a “read head” that hovers over elements in the data structure. The diagram below shows such a device

reading a string. Here the automata is like a scanner. It scans the string in one direction and

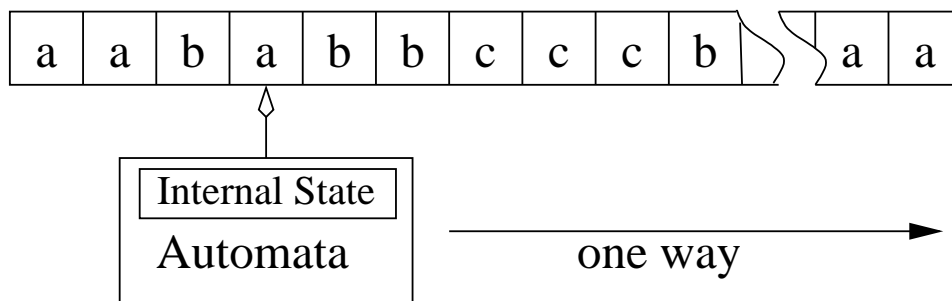


Figure 7.1: One-way automata

its internal state changes in response to new inputs. As Engelfriet and Hoogeboom (1999) explain, “Here the point of view has changed: the input is not fed into the automaton (like money into a coffee machine), but the automaton walks on the input string (like a mouse in a maze).”

The automata below is “Two-way.” This means it can move back and forth along the input tape.

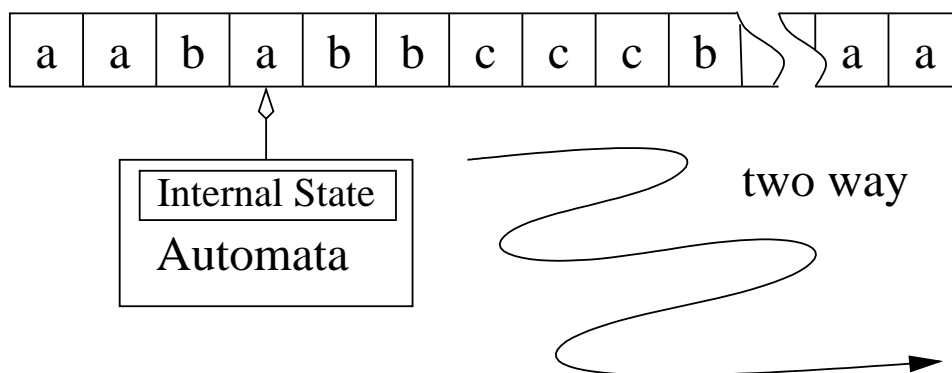


Figure 7.2: Two-way automata

Formally, for deterministic automata, the transition function  $\delta$  has as its domain  $Q \times \Sigma^*$  and as its co-domain  $\{\text{Left}, \text{Stay}, \text{Right}\} \times Q$ . The values **Left**, **Stay**, **Right** indicates whether the scanning device should move one position to the left or right or whether it should stay in its current position. If the scanner goes off the left edge, gets stuck in a loop, it rejects the string. It only accepts the string if it eventually moves off the right edge of the string

It is also possible to define non-deterministic two way automata.

In the case of string recognizers, all combinations are equally expressive.

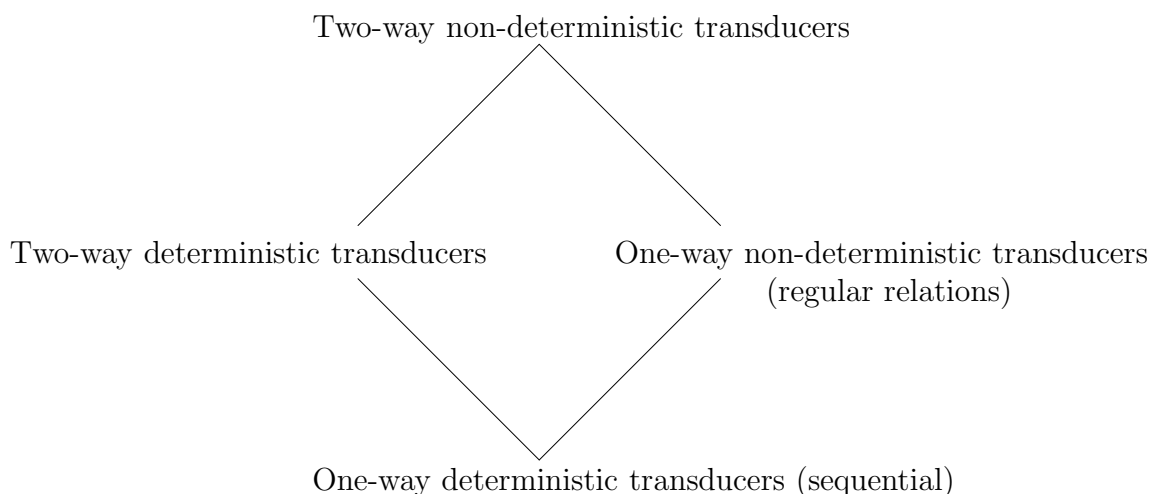
**Theorem 23.** *The class of stringsets recognized by*

1. *one-way deterministic acceptors,*

2. *two-way deterministic acceptors,*
3. *one-way non-deterministic acceptors, and*
4. *two-way non-deterministic acceptors*

are the same. They are the regular stringsets.

However, when we get to transducers, the picture changes. As mentioned, the deterministic/non-deterministic distinction matters for the transducer case. So does the one-way/two-way distinction. There are four distinct classes (Filiot and Reynier, 2016).



One reason to be interested in two-way automata is that they provide a natural way to describe reduplication in natural languages! Total reduplication patterns *cannot* be described with one-way string transducers. Your classmate Hossep Dolatian has shown this is the case for over a hundred reduplication patterns in the world’s languages and he will present this research at an upcoming linguistics conference at the University of Chicago.

For tree automata, there is an analog of “Two-way.” It is called “Walking” and these automata can move from one node to another (up to a parent or down to one of the children). They have been difficult to analyze; see Engelfriet and Hoogeboom (1999).

### 7.1.2 Natural Language Processing

Finite-state automata are widely used in natural language processing tasks (Roche and Schabes, 1997; Mohri, 1997, 2005; Jurafsky and Martin, 2008b). Many of these models rely on probabilities. Essentially, the transitions of the automata are weighted in the way we considered when the outputs of the sequential transducers were elements of a monoid.

In the domain of syntax, for many years people used context-free grammars or context-sensitive, but a recent trend has been to replace them with weighted tree automata (Knight and May, 2009; Maletti, 2009).

There are many aspects of NLP that this class did not cover since it focused on theoretical and foundational aspects. However, I hope that if you choose to further study NLP, the material you would have learned in this class would help.

### 7.1.3 Combining Automata and complex generalizations

Why do we care about the closure properties of automata? We care because they tell us that we can build new automata from old ones and that the new ones we build will behave the way we intend. This is especially useful if we have complex linguistic generalizations.

For example, consider a language which places stress on the first heavy syllable in a word if there is a heavy syllable. On words with no heavy syllables, stress goes on the last syllable. Simplifying we can assume that the letters are L and H for “Light” and “Heavy” respectively. Thus we have:

$$\begin{aligned} HLL &\mapsto \acute{H}LL \\ HHH &\mapsto \acute{H}HH \\ LLH &\mapsto LL\acute{H} \\ LLL &\mapsto LL\acute{L} \\ LLLL &\mapsto LLL\acute{L} \end{aligned}$$

It may be difficult to write a single automaton for this. However, if we are writing non-deterministic transducer, we can write one transducer for all the words with no heavy syllables (below left) and one transducer for the words with at least one heavy syllable (below right). Each of these is simple to write.



Next, since regular relations are closed under union we can simply take the union of these two machines. That’s it! Furthermore, if there is a deterministic transducer able to describe this pattern, there are algorithms that can determinize it (Mohri, 1997).

More generally, operations like intersection, composition, and concatenation let us describe complex generalizations in terms of simpler parts.

### 7.1.4 Subregular Classes of Automata

There are many types of regular stringsets, which can be further classified. The figure below shows one classification based on logic. Each of these classes have multiple characterizations in terms of logic, automata, and algebraic properties (McNaughton and Papert, 1971; Thomas, 1982; Rogers and Pullum, 2011; Rogers *et al.*, 2010). Rogers *et al.* (2013) argue

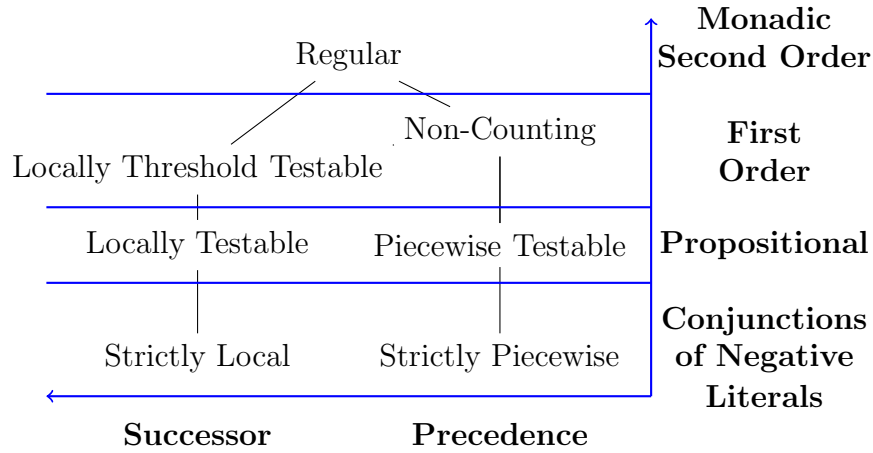


Figure 7.3: Subregular hierarchies of stringsets

that each of these classes correspond to a kind of model of memory with implications for cognition.

Heinz (2010) argues that all human phonotactic generalizations belong to the smallest classes shown here: Strictly Local and Strictly Piecewise generalizations. Heinz and Idsardi (2011, 2013) develop this view in the broader context of language and cognition.

What about subregular string transformations? This is much less well-studied and the University of Delaware has been at the forefront of this with Jane Chandlee’s PhD work (Chandlee, 2014; Chandlee *et al.*, 2014, 2015).

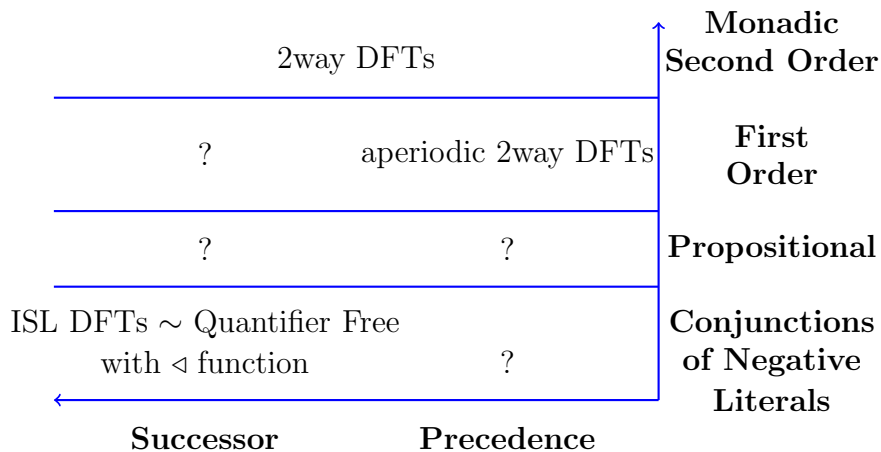


Figure 7.4: Subregular hierarchies of string-to-string functions

### 7.1.5 Learning Automata

Finally, there has been a lot of work on how automata can be learned. Here are some of the main results. Readers are referred to the following texts for more information Heinz *et al.* (2015); de la Higuera (2010); Heinz and Sempere (2016).

(Note that “identification in the limit” is a well-known definition for learning dating to Gold’s seminal work Gold (1967) even if it is a little controversial (Clark and Lappin, 2011). See also Heinz (2016).)

**Theorem 24.** *The regular stringsets cannot be identified in the limit from positive data.*

**Theorem 25.** *The regular stringsets and recognizable treesets can be efficiently identified in the limit from positive and negative examples.*

**Theorem 26.** *The Strictly  $k$ -Local, Strictly  $k$ -Piecewise, Locally  $k$ -Testable, Piecewise  $k$ -Testable, and Locally  $t$ -Threshold  $k$ -Testable classes (stringsets) each be efficiently identified in the limit from positive and negative data.*

Furthermore, there are concrete senses in which the lower classes are more efficiently learnable than the higher ones. I don’t know of work on subregular treesets, but these results should carry through.

**Theorem 27.** *The class of sequential functions is efficiently identifiable in the limit from positive data.*

**Theorem 28.** *The class of probability distributions over strings and over trees describable with weighted deterministic recognizers are efficiently approximable from positive data.*

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