MSO-definable Transductions over Word Models

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1 Introduction

These notes draw from [End72, RHF+13, RH14] and [EH01]. The idea here is to adapt the ideas of graph transdutions to word transductions where the words are represented with models.

2 Relational Models and Signatures

A n-ary *relation* is a relation of arity n. This means is expresses a relation among n different elements. So if D is the domain of elements then a n-ary relation over D is a subset of

$$D^n = \underbrace{D \times D \times \ldots \times D}_{n \text{ times}} .$$

For example a unary relation is a subset of D and a binary relation is a subset of $D \times D$.

A *relational model* is a structure. The structure is the information that exists about an object. The object can be identified as elements of a domain and the relationships among those elements. Sometimes these are called relational structures.

If the analyst has a class of objects in mind (for example words) then it is important to ensure that each unique object has some model and that distinct objects have distinct models.

The *signature* of a model defines a class of logically possible structures. It can be thought of as expressing the *type* of a model. At a minimum, a model signature contains a domain and one unary relation. At a a maximum, a model signaure contains a domain and a *finite number* of relations, which can be of various arities. Let

$$\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{R} \rangle$$

where there exists $n \in \mathbb{N}$ such that $\mathfrak{R} = \{R_i \mid (\forall\, 1 \leq i \leq n)[\exists\, a_i \in \mathbb{N} \wedge R_i \text{ is of arity } a_i]\}$. In words, $\mathfrak{R}$ is a set of n relations, and each R_i is a a_i-ary relation over $\mathfrak{D}$. We will denote the elements of $\mathfrak{R}$ with R_{i,a_i} so its arity is 'on its sleeve' so to speak. If i is understood from context, we will just write R_a.

As an example, let us consider word models. Fix an alphabet Σ. Then a word model has $|\Sigma|$ unary relations, one for each letter of the alphabet, and one binary relation, which is the ordering relation.

Some auxilliary concepts, such as *the interpretations of these relations*, are not part of the signature. The signature is purely a syntactic concept. The successor model and the precedence model have the same signature. They both contain $|\Sigma|$ unary relations and a singlw binary relation.

However, the two models are clearly different in how the binary relations are interpreted. For a domain $\mathfrak{D}$ in the successor model, we require $\triangleleft = \{(i, i+1) \mid i, i+1 \in \mathfrak{D}\}$ but for a domain $\mathfrak{D}$ in the precedence model we require $< \; = \{(i,j) \mid i,j \in \mathfrak{D} \wedge i < j\}$. These statements belong to how we interpret the signatures. Such statements can be considered part of a model in a broad sense, but they are not part of the model signature.

When we write models with a signature $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{R} \rangle$, we will write them as follows $\mathcal{M} = \langle \mathcal{D}, \mathcal{R} \rangle$ where there are n relations R_{i,a_i} in $\mathcal{R}$. Again we often write R_a for clarity, with the understanding that the arity a depends on the particular relation R (because it depends on i).

3 MSO Logic for relational models

Definition 1 (Sentences of MSO logic) *We consider a signature $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{R} \rangle$ with n relations in $\mathfrak{R}$.*

For all $x, y \in \{x_0, x_1, \ldots\}$, $X \in \{X_0, X_1, \ldots\}$, and for all signatures the following are sentences of MSO logic.

- $\boldsymbol{x = y}$ ***(equality)***
- $\boldsymbol{x \in X}$ ***(membership)***
- *For each $R_a \in \mathfrak{R}$:* $\boldsymbol{R_a(x_1, x_2, \ldots, x_a)}$ ***(atomic relational formualae)***

Also, if φ, ψ are sentences of MSO logic, then so are

- $(\neg \boldsymbol{\varphi})$ ***(negation)***
- $(\boldsymbol{\varphi} \vee \boldsymbol{\psi})$ ***(disjunction)***
- $(\exists \boldsymbol{x})[\varphi]$ ***(existential quantification for individuals)***
- $(\exists \boldsymbol{X})[\varphi]$ ***(existential quantification for sets of individuals)***

Nothing else is a sentence of MSO logic.

It is convenient to define additional syntax (whose intended meanings will follow from the semantics defined further below).

Definition 2 (Syntactic sugar) *If φ, ψ are sentences of MSO logic, then so are*

- $(\boldsymbol{\varphi \rightarrow \psi}) \stackrel{def}{=} ((\neg\boldsymbol{\varphi}) \vee \boldsymbol{\psi})$ (***implication***)
- $(\boldsymbol{\varphi \wedge \psi}) \stackrel{def}{=} (\neg((\neg\boldsymbol{\varphi}) \vee (\neg\boldsymbol{\psi})))$ (***conjunction***)
- $(\boldsymbol{\varphi \leftrightarrow \psi}) \stackrel{def}{=} ((\boldsymbol{\varphi \rightarrow \psi}) \wedge (\boldsymbol{\psi \rightarrow \varphi}))$ (***biconditional***)
- $(\forall \boldsymbol{x})[\boldsymbol{\varphi}] \stackrel{def}{=} (\neg(\exists \boldsymbol{x})[\neg\boldsymbol{\varphi}])$ (***universal quantification for*** ***individuals***)
- $(\forall \boldsymbol{X})[\boldsymbol{\varphi}] \stackrel{def}{=} (\neg(\exists \boldsymbol{X})[\neg\boldsymbol{\varphi}])$ (***universal quantification for sets of individuals***)

We assume familiarity with bound and free variables in MSO formulae. Let $MSO(\mathfrak{M})$ represent the set of formulae MSO formulae over the relational model

In order to interpret whether a model $\mathcal{M}$ with signature $\mathfrak{M}$ satisfies, or models, a sentence $\varphi \in MSO(\mathfrak{M})$ (written $\mathcal{M} \models \varphi$) variables must be assigned values. We assume an assignment function $\mathbb{S}$ which may be partial and which maps individual variables (like x) to individuals (elements of $\mathcal{D}$) and set-of-individual variables (like X) to sets of individuals (subsets of $\mathcal{D}$). If $\mathbb{S}$ maps a variable x to a element e it is denoted $\mathbb{S}[x \mapsto e]$ (and similarly $\mathbb{S}[X \mapsto S]$).

Then whether $\mathcal{M} \models \varphi$ is determined inductively.

Definition 3 (Interpreting sentences of MSO logic) *Here are the base cases. Note that many symbols (such as $=, \in, \vee$ and others) are used both syntactically and semantically. Care must be taken to ensure they are not confused. Here, and elsewhere, the syntactic expressions are in bold. Let $\mathcal{M} = \langle \mathcal{D}, \mathcal{R} \rangle$.*

$$\mathcal{M}, \mathbb{S}[x \mapsto e_1, y \mapsto e_2] \models \boldsymbol{x = y} \leftrightarrow e_1 = e_2$$

$$\mathcal{M}, \mathbb{S}[x \mapsto e, X \mapsto S] \models \boldsymbol{x \in X} \leftrightarrow e \in S$$

And for $R_a \in \mathfrak{R}$:

$$\mathcal{M}, \mathbb{S}[x_1 \mapsto e_1, \ldots, x_a \mapsto e_a] \models \boldsymbol{R_a(x_1, \ldots, x_a)} \leftrightarrow (e_1, \ldots, e_a) \in R_a$$

And here are the inductive cases.

$$\mathcal{M}, \mathbb{S} \models (\neg\boldsymbol{\varphi}) \leftrightarrow \neg(\mathcal{M}, \mathbb{S} \models \boldsymbol{\varphi})$$

$$\mathcal{M}, \mathbb{S} \models (\boldsymbol{\varphi \vee \psi}) \leftrightarrow \mathcal{M}, \mathbb{S} \models \boldsymbol{\varphi} \vee \mathcal{M}, \mathbb{S} \models \boldsymbol{\psi}$$

$$\mathcal{M}, \mathbb{S} \models (\exists x)[\boldsymbol{\varphi}] \leftrightarrow (\exists i \in \mathcal{D})\big[\mathcal{M}, \mathbb{S}[x \mapsto i] \models \boldsymbol{\varphi}\big]$$

$$\mathcal{M}, \mathbb{S} \models (\exists X)[\boldsymbol{\varphi}] \leftrightarrow (\exists S \subseteq \mathcal{D})\big[\mathcal{M}, \mathbb{S}[X \mapsto S] \models \boldsymbol{\varphi}\big]$$

That's it!

It is often convenient to define new (syntactic) predicates. Here are some useful ones.

- `true` $\stackrel{\text{def}}{=} (\forall \boldsymbol{x})[\boldsymbol{x = x}]$ (**truth**)
- `false` $\stackrel{\text{def}}{=} \neg$`true` (**falsehood**)
- $\boldsymbol{x \neq y} \stackrel{\text{def}}{=} \neg(\boldsymbol{x = y})$ (**distinctness**)

4 Relational Model Transductions

This is inspired by [EH01, §4]. We would like to be able to define a transduction which maps structures obeying one signature to structures of another signature.

4.1 Definition

A deterministic MSO-definable transduction τ from models with signature $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{R} \rangle$ to models with signature $\mathfrak{M}^\diamond = \langle \mathfrak{D}^\diamond, \mathfrak{R}^\diamond \rangle$ is specified by the following formulas.

1. a domain formula $\boldsymbol{\varphi_{dom}} \in MSO(\mathfrak{M})$ with no free variables;
2. a nonempty set $C \subset \mathbb{N}$ of finite cardinality;
3. for each $c \in C$, a formula $\boldsymbol{\varphi^c_\diamond(x)} \in MSO(\mathfrak{M})$ with one free node variable; and
4. for each $R^\diamond_a \in \mathfrak{R}^\diamond$, and $(c_1, \ldots, c_a) \in C^a$, there is a relational formula $\boldsymbol{\varphi^{c_1,\ldots,c_a}_{R^\diamond_a}(x_1, \ldots, x_a)} \in MSO(\mathfrak{M})$ with a free node variables.

For every $\mathcal{M} \models \boldsymbol{\varphi_{dom}}$ with domain $\mathcal{D}$, the image $\tau(\mathcal{M})$ is the structure $(\mathcal{D}^\diamond, \mathcal{R}^\diamond)$ defined as follows. (Let e^c stand for $(e, c) \in \mathcal{D} \times C$.)

- $\mathcal{D}^\diamond = \{e^c \mid e \in \mathcal{D}, c \in C, \mathcal{M} \models \boldsymbol{\varphi^c_\diamond(x)}\}$.
- For each $R^\diamond_a \in \mathcal{R}^\diamond$ and $(x_1^{c_1}, \ldots, x_a^{c_a}) \in \underbrace{\mathcal{D}^\diamond \times \ldots \times \mathcal{D}^\diamond}_{a \text{ times}}$, let $(x_1^{c_1}, \ldots, x_a^{c_a}) \in R^\diamond_a$ iff $\mathcal{M} \models \boldsymbol{\varphi^{c_1,\ldots,c_a}_{R^\diamond_a}(x_1, \ldots, x_a)}$

Informally, here is how this works.

- If element e in input model $\mathcal{M}$ satisfies $\varphi^c_\diamond(x)$ then node e^c in $\tau(\mathcal{M}) = \mathcal{M}^\diamond$ exists. It is an element of $\mathcal{D}^\diamond$. More formally: $\mathcal{M}, \mathbb{S}[x \mapsto e] \models \varphi^c_\diamond(x) \rightarrow e^c \in \mathcal{D}^\diamond$.
- Consider a unary relation $R^\diamond \in \mathcal{R}$, $c \in C$, and $e \in \mathcal{D}$. If $\mathcal{M}, \mathbb{S}[x \mapsto e] \models \boldsymbol{\varphi^c_{R^\diamond}(x)}$ then $e^c \in R^\diamond$. Informally, $e^c \in \mathcal{D}^\diamond$ has property $R^\diamond$ only if $\mathcal{M}, \mathbb{S}[x \mapsto e] \models \boldsymbol{\varphi^c_{R^\diamond}(x)}$.
- Consider a binary relation $R^\diamond \in \mathcal{R}$, $c_1, c_2 \in C$, and $e_1, e_2 \in \mathcal{D}$. If $\mathcal{M}, \mathbb{S}[x \mapsto e_1, y \mapsto e_2] \models \boldsymbol{\varphi^{c_1,c_2}_{R^\diamond}(x, y)}$ then $(e_1^{c_1}, e_2^{c_2}) \in R^\diamond$. Informally, elements $e_1^{c_1}, e_2^{c_2} \in \mathcal{D}^\diamond$ stand in the $R^\diamond$ relation only if $\mathcal{M}, \mathbb{S}[x \mapsto e_1, y \mapsto e_2] \models \boldsymbol{\varphi^{c_1,c_2}_{R^\diamond}(x, y)}$.

References

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