Chapter 1

Models and Logic

1.1 Strings and Stringsets

Assume a finite set of symbols. Traditionally, this set is denoted Σ. We also assume a non-commutative operation called *concatenation* with λ as the *identity* element. So for all strings u: $u \cdot \lambda = \lambda \cdot u = u$.

Strings are defined inductively:

1. Base case: λ is a string.
2. Inductive case: If u is a string and $\sigma \in \Sigma$ then $u \cdot \sigma$ is a string.

We refer to all strings with the notation Σ^*.

A *stringset* (=formal language) is a (possibly infinite) subset of Σ^*.

1.2 Word Models

We use the word 'word' synonymously with 'string.'

- A *model* of a word is a representation of it.
- A relational model contains two kinds of elements.
  A domain. This is a finite set of elements.
  Some relations over the domain elements.
- Guiding principles:
  1. Every word has some model.
  2. Different words must have different models.

Let $\Sigma = \{a, b, c\}$ and suppose we wish to model strings in Σ^*. Here is one model for it.

$$\mathbb{W}^{\triangleleft} = \langle \mathcal{D} \mid \triangleleft, P_a, P_b, P_c \rangle$$

$\mathcal{D}$	—	Finite set of elements (positions)
$\triangleleft$	—	A binary relation encoding immediate linear precedence on $\mathcal{D}$
P_σ	—	Unary relations (so subsets of $\mathcal{D}$) encoding positions at which σ occurs

Consider the string *abbab*.
The model of *abbab* under the signature $\mathbb{W}^\triangleleft$ (denoted $\mathcal{M}^\triangleleft_{abbab}$) looks like this.

$$\mathcal{M}^\triangleleft_{abbab} = \left\langle \begin{array}{l} \{1,2,3,4,5\}, \\ \{(1,2),(2,3),(3,4),(4,5)\}, \\ \{1,4\}, \\ \{2,3,5\}, \\ \emptyset \end{array} \right\rangle$$

Exercise 1.

1. Give models for these strings.
   (a) *abc*
   (b) *cacaca*
   (c) λ
2. Suppose we removed the unary relations from the signature so the it looks like this: $\mathbb{W}^\dagger = \langle \mathcal{D} \mid \triangleleft \rangle$. Can models with such a signature distinguish all strings in Σ^*?
3. Suppose we removed the successor relation from the signature so it looks like this: $\mathbb{W}^\ddagger = \langle \mathcal{D} \mid \ P_a, P_b, P_c \rangle$. Can models with such a signature distinguish all strings in Σ^*?

Generally, for any string $w = \sigma_1 \ldots \sigma_n \in \Sigma^*$, its word model with successor will be the following.

- The domain $\mathcal{D} := \{1 \ldots n\}$.
- The successor relation $\triangleleft := \ \{(i, i+1) \mid 1 \leq i < n\}$.
- For each $\sigma \in \Sigma^*$, we have $P_\sigma := \{i \in \mathcal{D} \mid \ \sigma_i = \sigma\}$.

Stepping back a bit, What is a word model? It is a representation of the word. It identifies sufficient *information* which distinguishes the word from other words. Furthermore, it says this information is *basic*.

From a computational perspective, this is the information that can be accessed immediately and obtained without any computational effort. For instance in the word *sotoʃ*, under the succesor model we can immediately know that the successor the s is the o because $1 \in P_s$ and $2 \in P_o$ and $(1,2) \in \triangleleft$. ALl that information is easy to look up. On the other hand, we *cannot* immediately know under the succesor model whether ʃ follows the *s*. This information cannot be looked up in the model (e.g. $(1, 5) \notin \triangleleft$). Instead one can deduce that ʃ follows the *s* but that requires some calculations. The logical perspective which will be explained as we go helps us understand the cost of that computation.

From a psychological perspective, the basic information can be said to make a claim about psychological reality. It says that the way words are represented in the mind is given by this basic information.

Exercise 2.

1. We can make models for any kind of object. Provide a model for trees. Make sure to follow the guiding principles.
2. In fact, there can be more than one model for a type of object. Provide a different word model for strings. (Hint: change how order is represented.)
3. Phonological theories often uses features as representational elements, not segments. How could you define a signature for a model that refers to features?

1.2.1 Sub-structures

Informally, a word w is a *sub-structure* of a word v we can "see" the structure w "inside" the structure v.

More formally, given a signature with a set $\mathcal{R}$ of binary and unary relations, a word w is a *sub-structure* of a word v if

- there exists an injection $f : \mathcal{D}_w \to \mathcal{D}_v$ such that
  1. the image of the domain of w is a *subset* of domain of the v
     - for each $x \in \mathcal{D}_w$, $f(x) \in \mathcal{D}_v$
  2. for each $R \in \mathcal{R}$, the image of R_w is a *subset* of R_v
     - for each unary R in the signature, and each $x \in \mathcal{D}_w$, if $x \in R_w$ then $f(x) \in R_v$.
     - for each binary R in the signature, and each $x, y \in \mathcal{D}_w$, if $(x, y) \in R_w$ then $(f(x), f(y)) \in R_v$.

We call f the *renaming function*. Can we *rename* elements in the domain of w so that it exists inside v?

(Note: the sub-structure relation can only be defined with "iff" instead of "if…then" as we have done above. When it matters to distinguish these, I will refer to the definition above as "implicational substructures".)

Example 1. *abc* is a sub-structure of *bbabcaa*. Rename as follows.
Let f(1) = 3, f(2) = 4, and f(3) = 5.

abc
$\mathcal{D} = \langle 1, 2, 3 \rangle$

$\lhd \;=$
$\{(1,2), (2,3)\}$

$P_a = \{1\}$

$P_b = \{2\}$

$P_c = \{3\}$

bbabcaa
$\mathcal{D} = \langle 1, 2, \mathbf{3}, \mathbf{4}, \mathbf{5}, 6, 7 \rangle$

$\lhd \;=$
$\{(1,2), (2,3), (\mathbf{3}, \mathbf{4}), (\mathbf{4}, \mathbf{5}), (6,7)\}$

$P_a = \{\mathbf{3}, 6, 7\}$

$P_b = \{1, 2, \mathbf{4}\}$

$P_c = \{\mathbf{5}\}$

Exercise 3. Consider the word model with successor.

$$\mathbb{W}^{\lhd} = \langle \mathcal{D} \mid \lhd, P_a, P_b, P_c \rangle$$

1. Explain why *aaa* is a sub-structure of *caaac*.
2. Explain why *abc* is not a sub-structure of *bbaccaa*.
3. Explain why *aba* is not a sub-structure of *abbabbaab*.
4. For any word w, is w is sub-structure of itself?
5. For any word w, if u is sub-structure of w and w is sub-structure of v, is u a sub-structure of v?

1.2.2 Substrings

Remember we distinguished strings from models of strings.

- String w is a substring of v if there exist strings $x, y \in \Sigma^*$ such that $v = xwy$.

Theorem 1. *Under the successor model, w is a substring of v iff the model of w is a sub-structure of the model of v.*