1 Propositional Logic

Recall the syntax and semantics of propositional logic.

Propositional Logic Assume a countable set of atomic propositions $\mathbb{P} = \{p_1, p_2, \ldots\}$ We often use letters such as p, q, and r to indicate atomic propositions from this set.

Syntax Each $p \in P$ is a sentence of propositional logic. If α, β are sentences of propositional logic, then so are $\neg\alpha$ and $(\alpha \wedge \beta)$. The set of sentences of propositional logic are denoted PROP($\mathbb{P}$).

Semantics A **valuation** is a total function from $v : \mathbb{P} \rightarrow \{\texttt{True}, \texttt{False}\}$. For each sentence of propositional logic φ, the interpretation of φ according to v, denoted $[\![\varphi]\!](v)$, is determined according to one of the following three cases.

1. There exists $p \in \mathbb{P}$ such that $\varphi = p$. Then $[\![\varphi]\!](v) = [\![p]\!](v) = v(p)$.
2. There exists $\alpha \in \text{PROP}(\mathbb{P})$ such that $\varphi = \neg\alpha$. Then $[\![\varphi]\!](v) = [\![\neg\alpha]\!](v) = \neg [\![\varphi]\!](v)$.
3. There exists $\alpha, \beta \in \text{PROP}(\mathbb{P})$ such that $\varphi = \alpha \wedge \beta$. Then $[\![\varphi]\!](v) = [\![\alpha \wedge \beta]\!](v) = \Big([\![\alpha]\!](v) \wedge [\![\beta]\!](v)\Big)$.

Today, we modify the semantics as follows. We let the atomic propositions denote *connected relational structures*. Models of words are (typically) connected relational structures but not all connected relational structures are models of words. We define a partial order ($\sqsubseteq$) among such structures. Then given a word w, and a sentence φ, we can check to see whether $\mathcal{M}_w \models \varphi$ as we did with FO logic. The valuation function v will essentially be given by $\mathcal{M}_w$. In particular, in the semantic interpretation of the atomic proposition, $[\![p]\!](v) = [\![p]\!](\mathcal{M}_w) = p \sqsubseteq \mathcal{M}_w$. In other words, if a connected relational structure is *contained within* $\mathcal{M}_w$ then $[\![p]\!](\mathcal{M}_w)$ is `True`; otherwise, it is `False`. The set of atomic propositions is then $\{S \mid \exists w \in \Sigma^*(S \sqsubseteq \mathcal{M}_w), S \text{ is connected}\}$.

2 Connected Relational Structures

What follows is adapted from, sometimes verbatim, from section 3 of [1].

We introduce a partial ordering over relational structures which conform to some model signature $\mathcal{M} := \langle D; \triangleleft, R_1, \ldots R_n \rangle$. To do so, we define the terms *connected, restriction*, and *factor*. For each structure $S = \langle D; \triangleleft, R_1, \ldots R_n \rangle$ let the binary "connectedness" relation C be defined as follows.

$$C \stackrel{\text{def}}{=} \big\{(x, y) \in D \times D \mid \exists i \in \{1 \ldots n\}, \exists (x_1 \ldots x_m) \in R_i, \exists s, t \in \{1 \ldots m\}, x = x_s, y = x_t\big\}$$

Informally, domain elements x and y belong to C provided they belong to some non-unary relation. Let C^* denote the symmetric transitive closure of C.

Definition 1 (Connected structure) *A structure $S = \langle D; \triangleleft, R_1, R_2, \ldots, R_n \rangle$ is* connected *iff for all $x, y \in D$, $(x, y) \in C^*$.*

For example, both $M^{\triangleleft}(abba)$ and $M^{<}(abba)$ below are connected structures. However, the structure $S_{ab,\ ba}$ shown below which is identical to $M^{\triangleleft}(abba)$ except it omits the pair (2,3) from the order relation is not connected since none of (1,3),(1,4), (2,3) nor (2,4) belong to C^*. For concreteness, $S_{ab,\ ba} = \big\langle D = \{1, 2, 3, 4\}; \triangleleft = \{(1, 2), (3, 4)\}, R_a = \{1, 4\}, R_b = \{2, 3\}, R_c = \emptyset \big\rangle$. Note that no string in Σ^* has structure $S_{ab,\ ba}$ as its model.

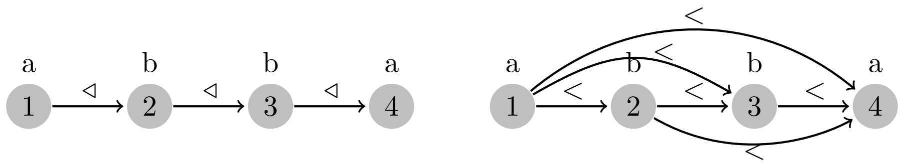

Figure 1: Visualizations of the successor (left) and precedence (right) models of *abba*.

Figure 2: Visualization of the structure $S_{ab,\ ba}$.

Definition 2 $A = \langle D^A; \triangleleft, R^A_1, \ldots, R^A_n \rangle$ is a restriction of $B = \langle D^B; \triangleleft, R^B_1, \ldots, R^B_n \rangle$ *iff* $D^A \subseteq D^B$ *and for each* m*-ary relation* R_i*, we have* $R^A_i = \{(x_1 \ldots x_m) \in R^B_i \mid x_1, \ldots, x_m \in D^A\}$.

Informally, one identifies a subset A of the domain of B and strips B of all elements and relations which are not wholly within A. What is left is a restriction of B to A.

Definition 3 *Structure* A *is a* subfactor *of structure* B *(*$A \sqsubseteq B$*) if* A *is connected, there exists a restriction of* B *denoted* B'*, and there exists* $h : A \to B'$ *such that for all* $a_1, \ldots a_m \in A$ *and for all* R_i *in the model signature: if* $h(a_1), \ldots h(a_m) \in B'$ *and* $R_i(a_1, \ldots a_m)$ *holds in* A *then* $R_i(h(a_1), \ldots h(a_m))$ *holds in* B'*. If* $A \sqsubseteq B$ *we also say that* B *is a* superfactor *of* A*.*

In other words, properties that hold of the connected structure A hold in a related way within B.

[1] go on to prove the following two lemmas.

Lemma 1 *For all strings* $x, y \in \Sigma^*$*,* x *is a substring of* y *iff* $M^{\triangleleft}(x) \sqsubseteq M^{\triangleleft}(y)$*.*

Lemma 2 *For all strings* $x, y \in \Sigma^*$*,* x *is a subsequence of* y *iff* $M^{<}(x) \sqsubseteq M^{<}(y)$*.*

Exercise 1

1. *Let* $\varphi = M^{\triangleleft}(abba)$*. What is* $[\![\varphi]\!]$*? How about* $[\![\neg\varphi]\!]$
2. *Let* $\varphi = M^{<}(abba)$*. What is* $[\![\varphi]\!]$*? How about* $[\![\neg\varphi]\!]$
3. *Consider a successor model with features. How would you express the constraint *NT?*
4. *Consider a precedence model with features. How would you express the constraint *N. . . L?*
5. *How could you express the conjunction of *NT and *CCC?*
6. *How could you express the conjunction of *NT and *N. . . L?*

3 Further reading

These notions are discussed in more detail in [1] and [2].

References

[1] Jane Chandlee, Remi Eyraud, Jeffrey Heinz, Adam Jardine, and Jonathan Rawski. Learning with partially ordered representations. In *Proceedings of the 16th Meeting on the Mathematics of Language*, pages 91–101, Toronto, Canada, 18–19 July 2019. Association for Computational Linguistics.

[2] James Rogers and Dakotah Lambert. Some classes of sets of structures definable without quantifiers. In *Proceedings of the 16th Meeting on the Mathematics of Language*, pages 63–77, Toronto, Canada, July 2019. Association for Computational Linguistics.