Syllable Structure and Model Theory (Strother-Garcia, 2019)

- rule-based account and OT models produce structures that are unattested in natural languages $\rightarrow$ we need another, more restrictive formalism

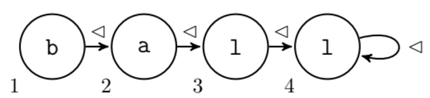
1 Word Model and Model Theories

- used for representing words (strings), where each letter is associated with a particular position
- **model theories** define classes of word models that share a common **signature**
- model theory $\mathfrak{M}$ has the signature $\langle \mathfrak{D}; \mathfrak{R}; \mathfrak{F} \rangle$, where:
  - $\mathfrak{D}$ is the set of positions:
    - unordered
    - functions and relations determine the positions
    - usually represented with a set of natural numbers
    - example: *ball*
      $\mathfrak{D} \stackrel{\text{def}}{=} \{1, 2, 3, 4\}$
  - $\mathfrak{R}$ is the set that includes a relation R_σ for every symbol σ in the alphabet Σ.
    - evaluated as Boolean expressions
    - example: *ball*
      $\Sigma = \{b, a, l\}$
      $\mathfrak{R} \stackrel{\text{def}}{=} \{R_b, R_a, R_l\}$, where R_i are unary relations
      $2 \in R_a$ or $R_a(2)$ or $a(2) \rightarrow$ position 2 is labeled as a, etc.
  - $\mathfrak{F}$ is the set of unary functions
    - map domain positions to domain positions
    - example: *ball*
      $\mathfrak{F} \stackrel{\text{def}}{=} \{pred(x), succ(x)\}$
      $succ(1) = 2$ or $1 \triangleleft 2$
    - **total** function yields output for every position of the domain; because $succ(x) = x + 1$, the final position in the string will also be its own successor ($succ(n) = n$)

1.1 The Successor Model Theory

$$\mathfrak{M}^{\triangleleft} \stackrel{\text{def}}{=} \langle \mathbb{N}; \{R_\sigma | \sigma \in \Sigma\}; \{pred(x), succ(x)\} \rangle$$

Visual representation of $\mathcal{M}^{\triangleleft}_{ball}$:



Word model:

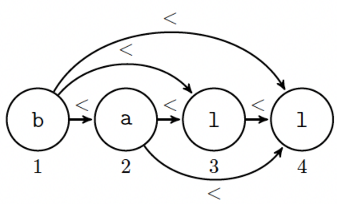
$$\mathcal{M}^{\triangleleft}_{ball} \overset{\text{def}}{=} \langle \mathcal{D}; \{R_a, R_b, R_l\}; \{pred(x), succ(x)\}\rangle$$
$$\mathcal{D} \overset{\text{def}}{=} \{1,2,3,4\}$$
$$R_a \overset{\text{def}}{=} \{2\}$$
$$R_b \overset{\text{def}}{=} \{1\}$$
$$R_l \overset{\text{def}}{=} \{3,4\}$$
$$succ(x) \overset{\text{def}}{=} \begin{cases} 2 & \Leftrightarrow x = 1 \\ 3 & \Leftrightarrow x = 2 \\ 4 & \Leftrightarrow x \in \{3,4\} \end{cases}$$
$$pred(x) \overset{\text{def}}{=} \begin{cases} 1 & \Leftrightarrow x \in \{1,2\} \\ 2 & \Leftrightarrow x = 3 \\ 3 & \Leftrightarrow x = 4 \end{cases}$$

1.2 The Precedence Model Theory

- the **domain** is a set of natural numbers
- unary relations and **general precedence relation** – two positions (binary relation) are ordered with respect to one another with a possibility of intervening positions, e.g. $R_<(x,y)$, $3 < 5$
- because of one-to-many relation, precedence cannot be encoded as a **function** $\rightarrow \mathfrak{F} \overset{\text{def}}{=} \varnothing$

$$\boxed{\mathfrak{M}^< \overset{\text{def}}{=} \langle \mathbb{N}; \{R_<, R_\sigma | \sigma \in \Sigma\}; \varnothing\rangle}$$

Visual representation of $\mathcal{M}^{<}_{ball}$:



Word model:

$$\mathcal{M}^{<}_{ball} \overset{\text{def}}{=} \langle \mathcal{D}; \{R_<, R_a, R_b, R_l\}; \varnothing\rangle$$
$$\mathcal{D} \overset{\text{def}}{=} \{1,2,3,4\}$$
$$R_< \overset{\text{def}}{=} \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}$$
$$R_a \overset{\text{def}}{=} \{2\}$$
$$R_b \overset{\text{def}}{=} \{1\}$$
$$R_l \overset{\text{def}}{=} \{3,4\}$$

2 Enriching Conventional Word Models

- **conventional** models are structured in a very simple way with an ordered sequence of alphabetic characters
- for linguistic purposes we would want to allow for more complex representation of a segment, e.g. phonological features

2.1 Enriching the Alphabet

- in a conventional model, sets of labeling relations for any two symbols are **disjoint**: each position belongs to a unary relation

$$\boxed{(\forall x, y \in \Sigma)[R_x \cap R_r = \varnothing]}$$

- we enrich the model by allowing more than one label per position
- it allows us to maintain similarities between segments, e.g. we want a model to be able to represent similarity between [b] and [p] such that both segments are are composed of [labial, stop] features bundle
- let $\mathcal{F}$ be a set of primitive features; and then $\Sigma = \mathcal{F}$

$$\boxed{\mathcal{F} \stackrel{\text{def}}{=} \{voice, cons, high, lab, alv, post, pal, vel, uv, phar, glot, stop, fric, nas, approx, lat\}}$$

- for each feature f in $\mathcal{F}$, there is a unary relation $R_f \in \mathfrak{R}$. Let $\mathcal{R}_f$ be the set of such relations

$$\boxed{\mathcal{R}_f \stackrel{\text{def}}{=} \{R_f | f \in \mathcal{F}\}}$$

  - example: *ball*

$$\boxed{\begin{array}{l} R_{voice}(x) \stackrel{\text{def}}{=} \{1\} \\ R_{labial}(x) \stackrel{\text{def}}{=} \{1\} \\ R_{stop}(x) \stackrel{\text{def}}{=} \{1\} \end{array}}$$

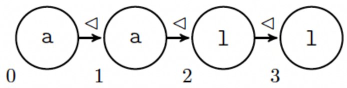
2.2 Enriching the Structure

- conventional models allow for only linear order whether it is general or immediate
- some linguistic representations, such as tone or syllable, require hierarchical representation which relies on **dominance** (which Jeff will talk about)

3 Graph Transductions

- in order to represent input-output mapping, we can use graphs transductions
- in order to map word model $\mathfrak{M}^A$ (input) to another word model $\mathfrak{M}^B$ (output) we must define a set of formulas, one for each relation R and function F in $\mathfrak{M}^B$

Visual representation of $\Gamma_{b \to a}(\mathrm{M}^{\triangleleft}_{ball})$:



Definition of the transduction $\Gamma_{b \to a}(\mathrm{M}^{\triangleleft}_{ball})$, where:

- w indicates an output position
- in the example below both input and output are share the same model theory, however it is not a requirement for such transductions

$$R_a(x^\omega) \stackrel{\text{def}}{=} R_a(x) \vee R_b(x)$$
$$R_b(x^\omega) \stackrel{\text{def}}{=} \texttt{FALSE}$$
$$R_l(x^\omega) \stackrel{\text{def}}{=} R_l(x)$$
$$succ(x^\omega) \stackrel{\text{def}}{=} succ(x)$$
$$pred(x^\omega) \stackrel{\text{def}}{=} pred(x)$$

4 Substructures

- in the example discussed ($ball$), we could also represent double l as a **substructure** if the model

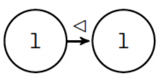
$$\boxed{\phi_{ll}(x) \stackrel{\text{def}}{=} l(x) \wedge l(succ(x))}$$

- in $\mathcal{M}^{\triangleleft}_{ball}$, $\phi_{ll}(x)$ is true for $x = 3$
- substructures are formalized as Existentially Quantified Conjunctions (EQCs)
- with the domain $\mathcal{D} = \{1, 2, ..., n\}$ and the set of relations $\mathcal{R} = \{R_1, R_2, ..., R_m\}$, an EQC is the conjunction of expressions that specify which positions belong to which relations

$$\boxed{(\exists x_1 \ldots, x_n)[\bigwedge_{i=1}^{m} R_i(x_a, \ldots, x_b)]}$$

- example:
  $\boxed{\phi_{ll}(x) \stackrel{\text{def}}{=} (\exists x, y)[l(x) \wedge l(succ(x)) \wedge x \triangleleft y]}$

Visual representation of the EQC $\phi_{ll}(x)$:



5 Formal Languages and Substructure Constraints

- we can use **formal languages** (set of well-formed strings) to represent SRs
- let $\mathcal{K}$ be the logical formula, Σ the alphabet, $\mathfrak{M}$ the model theory, w a word in Σ^*, and $\mathcal{M}_w$ the model of w. Then $\mathcal{K}$ defines a formal language $(L_{\mathcal{K}})$ which is a set of words in Σ^* whose word models satisfy. $\mathcal{K}$

$$L(\kappa) \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \mathcal{M}_w \models \kappa\}$$

- example:

  If expression:

  $$\psi_{\mathtt{d}} \stackrel{\text{def}}{=} (\exists x)[\mathtt{d}(x)]$$

  and a **substructure constraint** $\rightarrow$ conjunction of one or more literal. This could be though of as a well-formedness constraint:

  $$\kappa_{ball} \stackrel{\text{def}}{=} \psi_{\mathtt{ll}} \wedge \neg\psi_{\mathtt{d}}$$

  Then the language $L(\mathcal{K}_{ball})$ consists of word models that contain two consecutive *ll* and no *d*:

  $$L(\kappa_{ball}) = \{w \in \Sigma^* \mid \mathcal{M}_w^{\triangleleft} \models (\psi_{\mathtt{ll}} \wedge \neg\psi_{\mathtt{d}})\}$$

  – if N is a logical formula that is the conjunction of all substructure constraints needed to define a particular pattern, $L(N)$ is the set of all words generated by the **substructure constraint grammar** N.

6 User-defined Formulas

- x is b: $b(x) \stackrel{\text{def}}{=} voice(x) \wedge lab(x) \wedge stop(x)$
- x is an obstruent: $obs(x) \stackrel{\text{def}}{=} stop(x) \vee fric(x)$
- x is a sonorant: $son(x) \stackrel{\text{def}}{=} \neg obs(x)$
- positions:

$$init(x) \stackrel{\text{def}}{=} pred(x) = x$$
$$fin(x) \stackrel{\text{def}}{=} succ(x) = x$$
$$med(x) \stackrel{\text{def}}{=} \neg(init(x) \vee fin(x))$$