Syntax and Semantics of MSO and FO Logics

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1 Introduction

The difference between MSO and FO logic has to do with *quantification*. Both logics make use of *variables*. MSO makes us of two kinds of variables: variables that range over individual elements of the domain and variables that range over sets of individual elements of the domain. The former are denoted with lowercase letters such as x, y, z and the latter with uppercase letters X, Y, Z. While MSO uses both variables, FO logic only uses the former. It is literally those formulas of MSO logic *without* quantificaton over sets of individual elements of the domain. These notes draw from [End01, Hed04].

2 MSO Logic for relational models

Definition 1 (Formulass of MSO logic) *Fix a relational model signature $\mathbb{M}$ with finitely many relations $R \in \mathbb{M}$.*

The base cases. *Forall variables $x, y \in \{x_0, x_1, \ldots\}$, $X \in \{X_0, X_1, \ldots\}$, and for all relational structures the following are formulas of MSO logic.*

- $\boldsymbol{x = y}$ ***(equality)***
- $\boldsymbol{x \in X}$ ***(membership)***
- $\boldsymbol{R(\vec{x})}$ *for each $R \in \mathbb{M}$* ***(atomic relational formulas)***

It is understood that the $|\vec{x}| = \texttt{arity}(R)$. So if R is a unary relation, then $\vec{x} = (x)$. If R is a binary relation, then $\vec{x} = (x, y)$, and so on.

The inductive cases. *If φ, ψ are formulas of MSO logic, then so are*

- $(\neg\boldsymbol{\varphi})$ ***(negation)***
- $(\boldsymbol{\varphi} \vee \boldsymbol{\psi})$ ***(disjunction)***
- $(\exists \boldsymbol{x})[\varphi]$ ***(existential quantification for individuals)***
- $(\exists \boldsymbol{X})[\varphi]$ ***(existential quantification for sets of individuals)***

Nothing else is a formula of MSO logic.

It is convenient to define additional syntax (whose intended meanings will follow from the semantics defined further below).

Definition 2 (Syntactic sugar) *If φ, ψ are formulas of MSO logic, then so are*

- $(\boldsymbol{\varphi \rightarrow \psi}) \overset{def}{=} ((\neg\boldsymbol{\varphi}) \vee \boldsymbol{\psi})$ (***implication***)
- $(\boldsymbol{\varphi \wedge \psi}) \overset{def}{=} (\neg((\neg\boldsymbol{\varphi}) \vee (\neg\boldsymbol{\psi})))$ (***conjunction***)
- $(\boldsymbol{\varphi \leftrightarrow \psi}) \overset{def}{=} ((\boldsymbol{\varphi \rightarrow \psi}) \wedge (\boldsymbol{\psi \rightarrow \varphi}))$ (***biconditional***)
- $(\forall \boldsymbol{x})[\boldsymbol{\varphi}] \overset{def}{=} (\neg(\exists \boldsymbol{x})[\neg\boldsymbol{\varphi}])$ (***universal quantification for individuals***)
- $(\forall \boldsymbol{X})[\boldsymbol{\varphi}] \overset{def}{=} (\neg(\exists \boldsymbol{X})[\neg\boldsymbol{\varphi}])$ (***universal quantification for sets of individuals***)

In order to interpret whether a model $\mathcal{M}$ with signature $\mathbb{M}$ satisfies, or models, a formula $\varphi \in MSO(\mathbb{M})$ (written $\mathcal{M} \models \varphi$) variables must be assigned values. We assume an assignment function $\mathbb{S}$ which may be partial and which maps individual variables (like x) to individuals (elements of $\mathcal{D}$) and set-of-individual variables (like X) to sets of individuals (subsets of $\mathcal{D}$). If $\mathbb{S}$ maps a variable x to a element e it is denoted $\mathbb{S}[x \mapsto e]$ (and similarly $\mathbb{S}[X \mapsto S]$). Then whether $\mathcal{M} \models \varphi$ is determined inductively.

Definition 3 (Interpreting formulas of MSO logic) *Note that many symbols (such as $=, \in, \vee$ and others) are used both syntactically and semantically. Care must be taken to ensure they are not confused. Here, and elsewhere, the syntactic expressions are in bold.*

The base cases. *The third bullet is for each $R \in \mathbb{M}$.*

- $\mathcal{M}, \mathbb{S}[x \mapsto e_1, y \mapsto e_2] \models \boldsymbol{x = y} \leftrightarrow e_1 = e_2$
- $\mathcal{M}, \mathbb{S}[x \mapsto e, X \mapsto S] \models \boldsymbol{x \in X} \leftrightarrow e \in S$
- $\mathcal{M}, \mathbb{S}[\vec{x} \mapsto \vec{e}\,] \models \boldsymbol{R(\vec{x})} \leftrightarrow \vec{e} \in R$

The inductive cases.

- $\mathcal{M}, \mathbb{S} \models (\neg\boldsymbol{\varphi}) \leftrightarrow \neg(\mathcal{M}, \mathbb{S} \models \boldsymbol{\varphi})$
- $\mathcal{M}, \mathbb{S} \models (\boldsymbol{\varphi \vee \psi}) \leftrightarrow \mathcal{M}, \mathbb{S} \models \boldsymbol{\varphi} \vee \mathcal{M}, \mathbb{S} \models \boldsymbol{\psi}$
- $\mathcal{M}, \mathbb{S} \models (\exists \boldsymbol{x})[\boldsymbol{\varphi}] \leftrightarrow (\exists e \in \mathcal{D})\big[\mathcal{M}, \mathbb{S}[x \mapsto e] \models \boldsymbol{\varphi}\big]$
- $\mathcal{M}, \mathbb{S} \models (\exists \boldsymbol{X})[\boldsymbol{\varphi}] \leftrightarrow (\exists S \subseteq \mathcal{D})\big[\mathcal{M}, \mathbb{S}[X \mapsto S] \models \boldsymbol{\varphi}\big]$

The *free* variables of a formula φ are those variables in φ that are not quantified. A formula is a *sentence* if none of its variables are *free*. Only sentences can be interpreted.

Let Ω be a class of objects (like Σ^*). Let $\mathbb{M}$ denote a model signature for (elements of) Ω. Finally let φ be a sentence of MSO($\mathbb{M}$). Then the *extension* of φ is $[\![\varphi]\!] \overset{\text{def}}{=} \{\omega \in \Omega \mid \mathcal{M}_\omega \models \varphi\}$.

3 FO Logic

FO($\mathbb{M}$) is defined as all formulas of MSO logic without quantification over sets of indviduals. They are interpreted the same way as shown above.

References

[End01] Herbert B. Enderton. *A Mathematical Introduction to Logic*. Academic Press, 2nd edition, 2001.

[Hed04] Shawn Hedman. *A First Course in Logic*. Oxford University Press, 2004.